International Journal of Engineering Research And Management (IJERM)
ISSN : 2349- 2058, Volume-01, Issue-05, August 2014

Evaluating the Efficiency of DMUs with Fuzzy Data via FDH Model

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Abstract— **Data Envelopment Analysis (DEA) is a non-parametrical method for evaluating the efficiency of Decision Making Units (DMUs) using mathematical programming. The CCR model, the BCC model and the FDH model are well known as basic DEA models. These models based on the domination structure in primal form are characterized by how to determine the production possibility set from a viewpoint of dual form; the convex cone, the convex hull and the free disposable hull for the observed data, respectively.**
All these calculations occur when all data, that is the inputs and the outputs of Decision Making Units, are positive and crisp data. Now this question arises: if the data are Symmetric Triangular Fuzzy Number, how will be the method of computing the efficiency of Decision Making Units? In this article, we will introduce a method for evaluating the efficiency of Decision Making Units (DMUs) with FDH model when all data are symmetric triangular fuzzy numbers.
The basic idea is to transform the fuzzy model into a crisp mixed integer nonlinear programming problem by applying an $\alpha - cut$ approach. Finally, a numerical example is proposed to display the application of this method.

Index Terms— Data Envelopment Analysis, Free Disposable Hull, Mixed Integer linear programming, symmetrical triangular fuzzy number

I. INTRODUCTION

DEA is a powerful tool in estimating efficiency of decision making units with multiple inputs and outputs. Charnes, Cooper, Rhodes [1] were the pioneers of the field that introduced their first model named “CCR” in 1978. The original CCR model was applicable only to technologies characterized by constant return to scale globally. In what turned out to be major breakthrough, Banker, Charnes, and Cooper (BCC) [2], extended the CCR model to accommodate technologies that exhibit variable return to scale. In order to appear a single efficient unit in the reference set of inefficient DMUs, Deprins *et al.* [3], proposed a new type of DEA model called Free Disposal Hull (FDH) no convex model, mathematically the non-convex FDH model compares any DMU with an observed unit. This model has received a considerable amount of research attention. Tulkens [4] developed the methodological issues and applications of FDH model in retail banking, courts, and urban transit. Only a few researches have utilized fuzzy set theory to measure and evaluate efficiency performance. Sengupta [5] was the first to introduce a fuzzy mathematical programming approach where the constraints and objective function are not satisfied crisply. Fuzzy DEA models can more realistically represent real world problems than the conventional DEA models. We can consider two approaches for solving fuzzy CCR. The first one defuzzifies the fuzzy CCR model and changed it into the equivalent crisp model and the second one uses $\alpha - cut$ to create interval valued linear programming that solves the fuzzy DEA by parametric programming. . Tananka, Entani, Maeda [6], formulated two DEA models: one model that gives upper limit (best case) efficiency and one model that gives lower limit (worse case) efficiency. With defuzzification approach we first defuzzify the fuzzy inputs and outputs into crisp values, and then solve the resulting crisp model using an LP solver. Guo and Tanaka [7] considered the data as symmetrical triangular fuzzy vectors in fuzzy DEA model. After using $\alpha - cuts$ of constraints and comparison of intervals, they used a pair of linear programming problems to evaluate the efficiency of the DMU under consideration. In the most fuzzy CCR model after determining the $\alpha - cuts$ of objective function and constraints, the fuzzy triangular numbers are converted to crisp intervals. In most of the existing methods for possibilistic linear programming, where the $\alpha - cut$ is used, the solution is obtained by comparing the intervals in left and right hand side of constraints Shaocheng [8], Tanaka, Ichihashi and Asai [9]. Different methodologies have been suggested for comparison of the intervals In some of these methods simply the end point of the interval are considered for justification that makes the model very simple and hence a lot of information might have been lost. Saati

Manuscript received July 19, 2014
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and Memariani and Jahanshahloo [10], their method is to find a point in each interval, satisfying the set of constraints and at the same time maximizing the objective function. This is defined as a variable in the suggested method.
In this paper we want to evaluate the efficiency of DMUs with fuzzy inputs and outputs using FDH model. All of the data of the DMUs are fuzzy with symmetrical triangular membership function. We convert the fuzzy linear programming to interval linear programming by $\alpha-cut$ method. At each $\alpha-level$ the fuzzy inputs and outputs correspond to intervals, $(.^{m},.^{l},.^{u})$ .In this method, instead of comparing the equality of two intervals, a variable is defined in the interval, not only satisfies the set of constraints, but also maximizes the efficiency value.

II. FDH model with Fuzzy Data

Assume that there are *n* decision-making units (DMUs) to be evaluated, each DMU with *m* inputs and *s* outputs. Suppose that the inputs and outputs of DMU_j are respectively as $X_j=(x_{1j},x_{2j},\dots,x_{mj})$ and $Y_j=(y_{1j},y_{2j},\dots,y_{sj})$. FDH_{BCC}Model for evaluation of DMU_j is as follows:

$$
\begin{aligned}
&Min \quad \theta\\
&s.t.\ \sum_{j=1}^{n}\lambda_j x_{ij} \le \theta x_{io} \qquad i=1,2,\dots,m\\
&\sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{ro} \qquad r=1,2,\dots,s \qquad\qquad (1)\\
&\sum_{j=1}^{n}\lambda_j = 1\\
&\lambda_j \epsilon\{0,1\}
\end{aligned}
$$

The FDH_{BCC} model with fuzzy data can be written as:

$$
\begin{aligned}
&Min \quad \theta\\
&s.t.\ \sum_{j=1}^{n}\lambda_j \tilde{x}_{ij} \le \theta \tilde{x}_{io} \qquad i=1,2,\dots,m\\
&\sum_{j=1}^{n}\lambda_j \tilde{y}_{rj} \ge \tilde{y}_{ro} \qquad r=1,2,\dots,s \qquad\qquad (2)\\
&\sum_{j=1}^{n}\lambda_j = 1\\
&\lambda_j \epsilon\{0,1\}\ \ j=1,\dots,n
\end{aligned}
$$

Where,'$\sim$' indicates the fuzziness.
Now we consider that all of the inputs and outputs of DMUs are triangular fuzzy numbers. Let $\tilde{x}_{ij}=(x^m_{ij},x^l_{ij},x^u_{ij})$ and $\tilde{y}_{rj}=(y^m_{rj},y^l_{rj},y^u_{rj})$ which $(.^{l}),(.^{u})$ respectively indicates the lower bound and upper bound of fuzzy numbers. Therefore, the fuzzy FDH_{BCC} model is as follows:

$$
\begin{aligned}
&Min \quad \theta\\
&s.t.\ \sum_{j=1}^{n}\lambda_j (x^m_{ij},x^l_{ij},x^u_{ij}) \le \theta (x^m_{io},x^l_{io},x^u_{io}) \qquad i=1,2,\dots,m\\
&\sum_{j=1}^{n}\lambda_j \left(y^m_{rj},y^l_{rj},y^u_{rj}\right) \ge \left(y^m_{ro},y^l_{ro},y^u_{ro}\right) \qquad\qquad (3)\\
&\sum_{j=1}^{n}\lambda_j = 1\\
&\lambda_j \epsilon\{0,1\}\ \ j=1,\dots,n
\end{aligned}
$$

There are several methods to solve model (3).In most of these models for solving it using $\alpha-cut$, the intervals in both sides of the constraints are compared with each other. There are many methods for solving interval-programming problem. In this paper, instead of comparing intervals, we define variables in the intervals such that satisfy the set of constraints. To solve model (3) we apply concept of $\alpha-cut$. By introducing $\alpha-cuts$ of constraints the following model is obtained:

$$
\begin{aligned}
&Min \quad \theta\\
&s.t.\ \sum_{j=1}^{n}\lambda_j \left(\alpha x^m_{ij}+(1-\alpha)x^l_{ij},\alpha x^m_{ij}+(1-\alpha)x^u_{ij}\right) \le\\
&\theta\left(\alpha x^m_{io}+(1-\alpha)x^l_{io},\alpha x^m_{io}+(1-\alpha)x^u_{io}\right)\\
&\sum_{j=1}^{n}\lambda_j \left(\alpha y^m_{rj}+(1-\alpha)y^l_{rj},\alpha y^m_{rj}+(1-\alpha)y^u_{rj}\right) \ge\\
&\left(\alpha y^m_{ro}+(1-\alpha)y^l_{ro},\alpha y^m_{ro}+(1-\alpha)y^u_{ro}\right)\\
&\sum_{j=1}^{n}\lambda_j = 1 \qquad\qquad (4)\\
&\lambda_j \epsilon\{0,1\}\ \ j=1,\dots,n
\end{aligned}
$$

We can suppose the following variables at different $-cuts$:

$$
\begin{aligned}
&x^L_{ij}(\alpha)=\alpha x^m_{ij}+(1-\alpha)x^l_{ij}, \qquad j=1,\dots,n, i=1,\dots,m\\
&x^U_{ij}(\alpha)=\alpha x^m_{ij}+(1-\alpha)x^u_{ij} \qquad \forall\, i,j\\
&y^L_{rj}(\alpha)=\alpha y^m_{rj}+(1-\alpha)y^l_{rj}, \qquad r=1,\dots,s, j=1,\dots,n\\
&y^U_{rj}(\alpha)=\alpha y^m_{rj}+(1-\alpha)y^u_{rj}, \qquad \forall\, r,j
\end{aligned}
$$

By the above assumption, the following model is obtained:

$$
\begin{aligned}
&Min \quad \theta\\
&s.t.\ \sum_{j=1}^{n}\lambda_j (x^L_{ij}(\alpha),x^U_{ij}(\alpha)) \le \theta (x^L_{io}(\alpha),x^U_{io}(\alpha))\\
&\sum_{j=1}^{n}\lambda_j \left(y^L_{rj}(\alpha),y^U_{rj}(\alpha)\right) \ge \left(y^L_{ro}(\alpha),y^U_{ro}(\alpha)\right) \qquad (5)\\
&\sum_{j=1}^{n}\lambda_j = 1\\
&\lambda_j \epsilon\{0,1\}\ \ j=1,\dots,n
\end{aligned}
$$

Note that all the coefficients are stated as intervals. To solve the above interval programming we suppose $\hat{x}_{ij},\hat{y}_{rj}$ as follow:

$$
\begin{aligned}
&x^L_{ij}(\alpha) \le \hat{x}_{ij} \le x^U_{ij}(\alpha) \qquad i=1,\dots,m\\
&y^L_{rj}(\alpha) \le \hat{y}_{rj} \le y^U_{rj}(\alpha) \quad r=1,\dots,s
\end{aligned}
$$

By substituting the new variables, (5) can be written as follows:

$$
\begin{aligned}
&Min \quad \theta \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \le \theta \hat{x}_{io} \qquad i = 1,2,\dots,m \\
&\qquad \sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \ge \hat{y}_{ro} \qquad r = 1,2,\dots,s \qquad\qquad (6) \\
&\qquad \sum_{j=1}^{n} \lambda_j - 1 \\
&x_{ij}^{L}(\alpha) \le \hat{x}_{ij} \le x_{ij}^{U}(\alpha) \qquad i = 1,\dots,m \\
&y_{rj}^{L}(\alpha) \le \hat{y}_{rj} \le y_{rj}^{U}(\alpha) \qquad r = 1,\dots,s \\
&\qquad \lambda_j \epsilon \{0,1\} \quad j = 1,\dots,n
\end{aligned}
$$

This model is equivalent to a parametric programming, while $\alpha\epsilon(0,1]$ is a parameter. Thus, the fuzzy mixed integer linear programming problem given by (3) can be equivalent to a crisp parametric mixed integer linear programming problem.

III. Numerical example

As an example, consider the presented example in Guo and Tanaka (2001). Data are listed in table1 and each DMU consumes two symmetric triangular fuzzy inputs to produce two symmetric triangular fuzzy outputs. Fuzzy efficiency of DMUs with proposed model is shown in table 2.

DMU	D_1	D_2	D_3	D_4	D_5
$Input_1$	(4,3.5,4.5)	(2.9,2.9,2.9)	(4.9,4.4,5.4)	(4.1,3.4,4.8)	(6.5,5.9,7.1)
$Input_2$	(2.1,1.9,2.3)	(1.5,1.4,1.6)	(2.6,2.2,3)	(2.3,2.2,2.4)	(4.1,3.6,4.6)
$Output_1$	(2.6,2.4,2.8)	(2.2,2.2,2.2)	(3.2,2.7,3.7)	(2.9,2.5,2.3)	(5.1,4.4,5.8)
$Output_2$	(4.1,3.8,4.4)	(3.5,3.3,3.7)	(5.1,4.3,5.9)	(5.7,5.5,5.9)	(7.4,6.5,8.3)

Table1. Data set of DMUs

Application of model (6) for the data shown in table (1) at the different $\alpha - cuts$ by using Lingo software is shown in table (2).

DMU		A	B	C	D
$\alpha = 0.25$	Efficiency	0.4545	0.277	0.4545	0.333
$\alpha = 0.5$		1	0.6	0.6	0.428
$\alpha = 0.75$		1	0.777	1	0.538
$\alpha = 1$		1	1	1	0.666

Table2. Efficiency of DMUs at different $\alpha - cuts$

conclusion

FDH is a well-known mathematical programming approach which can be used to evaluate the efficiency and identify relationship between inputs and outputs of any decision making unit. The main assumption in the FDH model is that all of the inputs and outputs are crisp but in this paper we consider that all of the inputs and outputs are triangular fuzzy numbers.

Using fuzzy data, the model is converted to a possibilistic programming problem. We use Saati and Memariani method for converting this problem into a crisp linear programming based on $\alpha - cut$. In the Saati and memariani model they define suitable variables to solve. The substitutions of these variables make the model non-linear. By further suitable substitutions the model is linearized. Hence, by solving a linear programming problem for a given $\alpha - cut$, it is possible to generate a reliable and robust solution for possibilistic mathematical programming problems

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