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Free Vibration Analysis of Thick Cylindrical Composite Shells Using Higher Order Shear Deformation Theory

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Abstract— **This paper presents a free vibration analysis of laminated cylindrical shells using higher order shear deformation theory recently developed by Zannon et al., (2014). Equilibrium equations of motion, stress resultants and strain displacement relations are developed using Hamilton's principles. These equations are based on the theory of elasticity with terms truncated to a third order (hence, third order theory). The equations can be used to characterize the system under various boundary conditions. For the present analysis, the boundary conditions considered are simply supported with a cross-ply lamination sequence. Here, we specifically developed mathematical formulation that considers transverse normal stress, shear deformation and rotary inertia for the shell system. The free vibrational analysis using the third -order shears deformation shell theory led to a system of generalized eigenvalue problem. This eigenvalue problem is then solved numerically using commercial Matlab software to obtain the free undamped vibrational frequencies. Since the higher frequencies are often damped, the first five natural frequency parameters are reported and compared with previously published first order approximation and three dimensional finite element analyses.**

Index Terms— **Free Vibration, Hamilton Principles, Thick Shell, Natural Frequency, Third order shear deformation, Cross-ply, Eigenvalue**

I. INTRODUCTION

The Literature on shell vibrations is vast. Hundreds of papers were published on free vibration especially for cylindrical shells (Reddy, 2004; Reissner, 1945; Librescu & Frederick, 1989; Love, 1892; Leissa, 1993; Timoshenko & Woinowsky 1959; Koiter, 1969). Various engineering applications and developments are underway for laminated composite shells. For example, aerospace industry, material technology, marine, petroleum, construction and automotive engineering (Qatu et al., 2013; Qatu, 2004). The vibration of thick shells has conventionally been solved using the 1st order shear deformation shell theory (Qatu et al., 2013; Asadia et al., 2012; Chen et al., 1997; Love, 1892). Often three dimensional theory of elasticity is used for solving theories of shell structures. Thus, three dimensional analyses of shells is considered to be the most accurate, but is complex and time consuming when the shell system has many graded layers due to the incorporation of composite materials. Even for today's availability of computational facilities, three dimensional finite element analyses for most practical problems are not feasible. Lately, many shell structures are making use of the functionality of composite materials, which are often energy efficient and durable. Composite shells often consist of many layers of varied strength materials; therefore, each layer has different material characteristics (Naghdi & Berry, 1964; Soldatos, 1999; Noor et al., 1996; Koiter, 1969).

A comprehensive summary and discussion of shell theories using first order deformation and various mode shape characteristics has been done by researches (Leissa, 1993; Reissner, 1945; Kurylov & Amabili 2010; Qatu 2002; Amabili & Paidoussis 2003) over the past 40 years. The difficulties of composite materials make shell structure hard to explore three dimensional theory of elasticity, which is important for the development of various shell and plate theories. Even though, researchers have done an extensive study on various thin and thick shell vibrations using lower order approximation in displacements, which is one of drawback of many practical applications. Zannon et. al. (2014) remediated some of these drawbacks by incorporating higher order approximations and a non-zero mid-surface displacement in the shell theory. However, any new theory in vibrational analysis has to establish the basic fundamentals concerning free vibrations before we proceed to further analyses. Different plate theories were classified into classical theories; two of the most important ones are first order shear deformation theories and higher-order shear deformation theories based on the thickness ratio. In the case of thick shell theory, the effect of the thickness of shells and the depth ratio of shells should not be neglected (Librescu & Frederick, 1989; Leissa, 1993; Timoshenko & Woinowsky 1959; Zannon et al., 2014; Qatu, 2010; Asadi et al., 2012; Qatu et al., 2013). The applications of shells vibration analysis are very crucial for structural design. Furthermore, experimental design of such structures can be rather cumbersome and costly. Therefore, theoretical model becomes viable and should elucidate various modes of vibrations during the impact period. Hence, composite shell theories are used widely in many shell structural applications.

The characteristic properties of vibrational behavior of composite laminated shells are used to understand the stability criteria of the system as a whole. Over the past years many researchers have analyzed various aspects of vibrations analysis of the system.

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Nomenclature

$A_{ij}, \hat{A}_{ij}, \overline{A}_{ij}$ *stretching and shearing stiffness parameters*

$A_{ij_\alpha}, A_{ij_\beta}, A_{ij_n}$ *stiffness parameters*

$B_{ij}, \hat{B}_{ij}, \overline{B}_{ij}$ *coupling stiffness parameters*

$B_{ij_\alpha}, B_{ij_\beta}, B_{ij_n}$ *stiffness parameters*

$D_{ij}, \hat{D}_{ij}, \overline{D}_{ij}$ *bending and twisting stiffness parameters*

$D_{ij_\alpha}, D_{ij_\beta}, D_{ij_n}$ *stiffness parameters*

$\left.\begin{matrix} E_{ij}, F_{ij}, L_{ij}, E_{ij_\alpha}, E_{ij_\beta}, E_{ij_n} \\ F_{ij_\alpha}, F_{ij_\beta}, F_{ij_n}, L_{ij_\alpha}, L_{ij_\beta}, L_{ij_n} \end{matrix}\right\}$ *higher order stiffness parameters*

(α, β, z) *shell coordinates*

I_i *rotary inertia*

$\rho^{(k)}$ *mass density of kth layer*

$\sigma_\alpha, \sigma_\beta, \sigma_z$ *normal stress*

$\sigma_{\alpha\beta}, \sigma_{\beta z}, \sigma_{\alpha z}$ *shear stress*

$\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_z$ *normal strains*

$\gamma_{\alpha\beta}, \gamma_{\alpha z}, \gamma_{\beta z}$ *shear strains*

$\overline{Q}_{ij}^{(k)}, \overline{Q_{ij}}$ *elastic stiffness parameters for layer k*

Q_α, Q_β *transverse shear force*

$m_\alpha^{(1)}, m_\beta^{(1)}, m_\alpha^{(2)}, m_\beta^{(2)}$ *distributed couples*

q_z, q_β, q_α *distributed forces*

A, B *Lame′ parameters*

$\varphi_\alpha, \varphi_\beta$ *higher order terms rotation of transverse normal*

R_α, R_β *principle radii of curvature*

$\kappa_\alpha, \kappa_\beta$ *curvature values of the α and β curves*

$\kappa_{\alpha\beta}$ *twist curvature*

For example, Qatu (1994) studied the vibration of symmetrically laminated composite shells with different boundary conditions. Also, Noor (1990) and Leissa (1993) presented the solution and the vibrational analysis for symmetrically laminated thin cantilevered composite shells. Furthermore, Qatu (1994) analyzed the free vibrations of composite laminated thin shells. These researchers have utilized the classical shell theory in their analysis. Chen et al. (1997) studied the vibration of symmetrically laminated composite shells using the higher order shear deformation theory. However, not many researchers have undertaken the concepts of isotropic unsymmetrical laminated composite shells of free, undamped vibration problem (Aydo˘gdu & Timarci, 2007; Khare & Rode, 2005). Furthermore, there are some studies on unsymmetrical cross ply shells with very limited boundary conditions (Lee & Reddy, 2004; Zhou et al., 2002; Asadi et al., 2012; Naghdi & Berry, 1964; Reddy, 2004).

In order to fully understand the various vibrational characteristics of the composite cross- ply thick shell dynamics, the theory of first order shear deformation has to be modified. Towards this attempt, this part of the research has undertaken by incorporating the thickness-depth ratio of the thick shell. Therefore, basic equations derived for shells and the respective solution procedures often encounters difficulties due to the term $(1+\frac{z}{R})$ in both the stress resultant and strain displacement equations, especially for thick shell. However, some researchers have developed higher order shell theories by neglecting the $(1+\frac{z}{R})$ term (Reddy, 2004; Librescu & Frederick, 1989; Koiter, 1969; Noor et al., 1996), which is not applicable for thick shell structures. More specifically, Zannon et.al (2014) incorporated this term in the mathematical formulation of a third-order shear deformation shell thick theory by Zannon (TSDTZ) without violating the classical shell theory assumptions. The solution of the TSDTZ is used to obtain the free vibrational characteristics of composite thick shells with specified boundary conditions of simply supported cross-ply laminates. The computational results obtained from TSDTZ are encouraging. It showed that TSDTZ improved the results in comparison to the first order shear deformation theory suggested by several other researchers (Asadi et al., 2012; Qatu et al., 2012; Asadi & Qatu, 2012; Qatu, 1994, 2004).

II. Theories and Formulation

Basic equations of TSDTZ have been developed Zannon et al., (2014) to study the free vibrational analysis of simply supported cross-ply cylindrical shell. The analysis of shell type elastic body is based on three fundamental principles such as equilibrium, continuity (displacement) and constitutive relationship (stress-strain).

We used Hamilton's principle to include various physical mechanisms such as extensional motion (stress, strain, transvers shear), and rotatory inertia of complex shells which undergo resonance (Deana & Werbya, 1992; Naghdi & Berry, 1964). The Hamilton's principle (Khare & Rode, 2005; Qatu, 2004; Zannon et al., 2014) for the equations of motion of a body with surface S between two arbitrary time intervals t_0 and t_1 requires that

$$\delta \int_{t_0}^{t_1} (K+W-U)dt = 0 \,, \qquad (2.1)$$

where U is the shell strain energy, K the kinetic energy and W the external work by the system are given in appendix A. However, the approximation of displacement components using the third-order shear deformation shell theory can be written as Zannon et al., (2014)

$$\begin{aligned} u(\alpha,\beta,z) &= u_0(\alpha,\beta) + z\psi_\alpha(\alpha,\beta) + z^3\varphi_\alpha(\alpha,\beta) \\ v(\alpha,\beta,z) &= v_0(\alpha,\beta) + z\,\psi_\beta(\alpha,\beta) + z^3\varphi_\beta(\alpha,\beta) \ldots (2.2) \\ w(\alpha,\beta,z) &= w_0(\alpha,\beta) + z\psi_z(\alpha,\beta). \end{aligned}$$

Where h is the shell thickness and $\frac{-h}{2} \le z \le \frac{h}{2}$. u_0, v_0, w_0 are midsurface displacements of the shell and ψ_α, ψ_β, ψ_z are midsurface rotations and $\varphi_\alpha, \varphi_\beta$ are higher order terms rotation of transverse normal. The corresponding strain-displacement equations are Zannon et al., (2014):

$$\varepsilon_\alpha = \frac{1}{(1+ z/R_\alpha)}(\varepsilon_{0\alpha} + z\kappa_\alpha{}^{(1)} + z^2\kappa_\alpha{}^{(2)})$$

$$\varepsilon_\beta = \frac{1}{(1+ z/R_\beta)}(\varepsilon_{0\beta} + z\kappa_\beta{}^{(1)} + z^2\kappa_\beta{}^{(2)})$$

$$\varepsilon_z = \psi_z(\alpha,\beta) \neq 0$$

$$\varepsilon_{\alpha\beta} = \frac{1}{(1+ z/R_\alpha)}(\varepsilon_{0\alpha\beta} + z\kappa_{\alpha\beta}{}^{(1)} + z^2\kappa_{\alpha\beta}{}^{(2)})$$

$$\varepsilon_{\beta\alpha} = \frac{1}{(1+ z/R_\beta)}(\varepsilon_{0\beta\alpha} + z\kappa_{\beta\alpha}{}^{(1)} + z^2\kappa_{\beta\alpha}{}^{(2)}).$$

$$\gamma_{\alpha z} = \frac{1}{(1+ z/R_\alpha)}(\gamma_{0\alpha z} + zG^{(1)} + z^2G^{(2)})$$

$$\gamma_{\beta z} = \frac{1}{(1+ z/R_\beta)}(\gamma_{0\beta z} + zE^{(1)} + z^2E^{(2)}) \qquad (2.3)$$

and where

$$G^{(1)} = \frac{1}{A}\frac{\partial\psi_z}{\partial\alpha} + 2\phi_\alpha - \frac{\psi_\beta}{R_{\alpha\beta}}, \quad G^{(2)} = \frac{\phi_\beta}{R_\alpha} - \frac{\phi_\alpha}{R_{\alpha\beta}}.$$

$$E^{(1)} = \frac{1}{B}\frac{\partial\psi_z}{\partial\beta} + 2\phi_\beta - \frac{\psi_\alpha}{R_{\alpha\beta}}, \quad E^{(2)} = \frac{\phi_\beta}{R_\beta} - \frac{\phi_\alpha}{R_{\alpha\beta}}.$$

The functions on the right-hand side of Eq. (2.3) are

$$\varepsilon_{0\alpha} = \frac{1}{A}\frac{\partial u_0}{\partial\alpha} + \frac{v_0}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0}{R_\alpha}, \ \varepsilon_{0\beta} = \frac{1}{B}\frac{\partial v_0}{\partial\beta} + \frac{u_0}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0}{R_\beta},$$

$$\varepsilon_{0\alpha\beta} = \frac{1}{A}\frac{\partial v_0}{\partial\alpha} - \frac{u_0}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0}{R_{\alpha\beta}}, \ \varepsilon_{0\beta\alpha} = \frac{1}{B}\frac{\partial u_0}{\partial\beta} - \frac{v_0}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0}{R_{\alpha\beta}},$$

$$\gamma_{0\alpha z} = \frac{1}{A}\frac{\partial w_0}{\partial\alpha} - \frac{u_0}{R_\alpha} - \frac{v_0}{R_{\alpha\beta}} + \psi_\alpha, \ \gamma_{0\beta z} = \frac{1}{B}\frac{\partial w_0}{\partial\beta} - \frac{v_0}{R_\beta} - \frac{u_0}{R_{\alpha\beta}} + \psi_\beta, \qquad (2.4)$$

$$\kappa_\alpha^{(1)} = \frac{1}{A}\frac{\partial\psi_\alpha}{\partial\alpha} + \frac{\psi_\beta}{AB}\frac{\partial A}{\partial\beta} + \frac{\psi_z}{R_\alpha}, \ \kappa_\beta^{(1)} = \frac{1}{A}\frac{\partial\psi_\beta}{\partial\beta} + \frac{\psi_\alpha}{AB}\frac{\partial B}{\partial\alpha} + \frac{\psi_z}{R_\beta},$$

$$\kappa_\alpha^{(2)} = \frac{1}{A}\frac{\partial\phi_\alpha}{\partial\alpha} + \frac{\phi_\beta}{AB}\frac{\partial A}{\partial\beta}, \ \kappa_\beta^{(2)} = \frac{1}{A}\frac{\partial\phi_\beta}{\partial\beta} + \frac{\phi_\alpha}{AB}\frac{\partial B}{\partial\alpha},$$

$$\kappa_{\alpha\beta}^{(1)} = \frac{1}{A}\frac{\partial\psi_\beta}{\partial\alpha} - \frac{\psi_\alpha}{AB}\frac{\partial A}{\partial\beta} + \frac{\psi_z}{R_{\alpha\beta}}, \ \kappa_{\alpha\beta}^{(2)} = \frac{1}{A}\frac{\partial\phi_\beta}{\partial\alpha} - \frac{\phi_\alpha}{AB}\frac{\partial A}{\partial\beta}.$$

After integrating the stress over the thickness of shell by incorporating the term $(1+\frac{z}{R})$, we get the following moment and force resultants:

$$\begin{Bmatrix} N_\alpha \\ N_{\alpha\beta} \\ Q_\alpha \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1+ z/R_\beta\right) dz, \qquad (2.5)$$

$$\begin{Bmatrix} N_\beta \\ N_{\beta\alpha} \\ Q_\beta \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} \left(1+ z/R_\alpha\right) dz.$$

The bending and twisting moment resultants defined as Zannon et al., (2014):

$$\begin{Bmatrix} M_\alpha^{(1)} \\ M_{\alpha\beta}^{(1)} \\ P_\alpha^{(1)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1 + z/R_\beta\right) z\, dz,$$

$$\begin{Bmatrix} M_\beta^{(1)} \\ M_{\beta\alpha}^{(1)} \\ P_\beta^{(1)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} \left(1 + z/R_\alpha\right) z\, dz,$$

$$\begin{Bmatrix} M_\alpha^{(2)} \\ M_{\alpha\beta}^{(2)} \\ P_\alpha^{(2)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\alpha \\ \sigma_{\alpha\beta} \\ \sigma_{\alpha z} \end{Bmatrix} \left(1 + z/R_\beta\right) z^2\, dz,$$

$$\begin{Bmatrix} M_\beta^{(2)} \\ M_{\beta\alpha}^{(2)} \\ P_\beta^{(1)} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\beta \\ \sigma_{\alpha\beta} \\ \sigma_{\beta z} \end{Bmatrix} \left(1 + z/R_\alpha\right) z^2\, dz. \tag{2.6}$$

Where $P_\alpha^{(1)}, P_\alpha^{(2)}, P_\beta^{(1)}$ and $P_\beta^{(2)}$ are higher order shear resultants terms. The stress resultant or the stiffness matrices obtained from the above equations are given in Appendix B. Substituting these stiffness parameters in the Hamilton equation (2.1), and simplifying the resulting equations, we get the following equations of motion:

$$\frac{\partial}{\partial\alpha}(BN_\alpha) - \frac{\partial B}{\partial\alpha}N_\beta + \frac{\partial A}{\partial\beta}N_{\alpha\beta} + \frac{\partial}{\partial\beta}(AN_{\beta\alpha}) + \frac{AB}{R_\alpha}Q_\alpha + \frac{AB}{R_{\alpha\beta}}Q_\beta + ABq_\alpha = AB(\overline{I}_1 u_0 + \overline{I}_2\psi_\alpha).$$

$$\frac{\partial}{\partial\alpha}(BN_{\alpha\beta}) + \frac{\partial B}{\partial\alpha}N_{\beta\alpha} - \frac{\partial A}{\partial\beta}N_\alpha + \frac{\partial}{\partial\beta}(AN_\beta) + \frac{AB}{R_{\alpha\beta}}Q_\alpha + \frac{AB}{R_\beta}Q_\beta + ABq_\beta = AB(\overline{I}_1 v_0 + \overline{I}_2\psi_\beta).$$

$$\frac{\partial}{\partial\alpha}(BQ_\alpha) + \frac{\partial}{\partial\beta}(AQ_\beta) - AB\left(\frac{N_\alpha}{R_\alpha} + \frac{N_\beta}{R_\beta} + \frac{N_{\alpha\beta} + N_{\beta\alpha}}{R_{\alpha\beta}}\right) + ABq_n = AB(\overline{I}_1 w_0).$$

$$\frac{\partial}{\partial\alpha}(BM_\alpha^{(1)}) - \frac{\partial B}{\partial\alpha}M_\beta^{(1)} + \frac{\partial A}{\partial\beta}M_{\alpha\beta}^{(1)} + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}^{(1)}) - ABQ_\alpha + \frac{AB}{R_{\alpha\beta}}P_\beta^{(1)} + ABm_\alpha^{(1)} = AB(\overline{I}_2 u_0 + \overline{I}_3\psi_\alpha).$$

$$\frac{\partial}{\partial\beta}(AM_\beta^{(1)}) - \frac{\partial A}{\partial\beta}M_\alpha^{(1)} + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}^{(1)}) + \frac{\partial B}{\partial\alpha}M_{\beta\alpha}^{(1)} - ABQ_\beta + \frac{AB}{R_{\alpha\beta}}P_\alpha^{(1)} + ABm_\beta^{(1)} = AB(\overline{I}_2 v_0 + \overline{I}_3\psi_\beta).$$

$$\frac{\partial}{\partial\alpha}(BP_\alpha^{(1)}) + \frac{\partial}{\partial\beta}(AP_\beta^{(1)}) - AB\left(N_z + \frac{M_\alpha^{(1)}}{R_\alpha} + \frac{M_\beta^{(1)}}{R_\beta} + \frac{M_{\alpha\beta}^{(1)}}{R_{\alpha\beta}} + \frac{M_{\beta\alpha}^{(1)}}{R_{\alpha\beta}}\right) + ABm_z = AB(\overline{I}_3\psi_z).$$

$$\frac{\partial}{\partial\alpha}(BM_\alpha^{(2)}) - \frac{\partial B}{\partial\alpha}M_\beta^{(2)} + \frac{\partial}{\partial\beta}(AM_{\beta\alpha}^{(2)}) - \left(2ABP_\alpha^{(1)} + \frac{AB}{R_{\alpha\beta}}P_\alpha^{(2)} + \frac{ABP_\beta^{(2)}}{R_{\alpha\beta}}\right) + ABm_\alpha^{(2)} = AB(\overline{I}_3 u_0 + \overline{I}_4\varphi_\alpha). \tag{2.7}$$

$$\frac{\partial}{\partial\beta}(AM_\beta^{(2)}) - \frac{\partial A}{\partial\beta}M_\alpha^{(2)} + \frac{\partial}{\partial\alpha}(BM_{\alpha\beta}^{(2)}) - \left(\frac{AB}{R_\beta}P_\beta^{(2)} + \frac{AB}{R_\alpha}P_\alpha^{(2)} + 2ABP_\beta^{(1)}\right) + ABm_\beta^{(2)} = AB(\overline{I}_3 v_0 + \overline{I}_4\varphi_\beta).$$

Where the rotary inertia is given as Zannon et al., (2014):

$$\overline{I}_i = \left(I_i + I_{i+1}\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)+\frac{I_{i+2}}{R_\alpha R_\beta}\right), \forall\ i=1,2,3,4 \tag{2.8}$$

The boundary conditions for S_2 are given below, where α is a constant

$$\begin{array}{llll}
either & N_{0\alpha}-N_\alpha=0 & or & u_0=0\\
either & N_{0\alpha\beta}-N_{\alpha\beta}=0 & or & v_0=0\\
either & Q_{0\alpha}-Q_\alpha=0 & or & w_0=0\\
either & M^{(1)}_{0\alpha}-M^{(1)}_\alpha=0 & or & \psi_\alpha=0\\
either & M^{(1)}_{0\alpha\beta}-M^{(1)}_{\alpha\beta}=0 & or & \psi_\beta=0\ .\\
either & P^{(1)}_{0\alpha}-P^{(1)}_\alpha=0 & or & \psi_z=0\\
either & M^{(2)}_{0\alpha}-M^{(2)}_\alpha=0 & or & \varphi_\alpha=0\\
either & M^{(2)}_{0\alpha\beta}-M^{(2)}_{\alpha\beta}=0 & or & \varphi_\beta=0
\end{array} \tag{2.9}$$

Similar boundary conditions are obtained by taking β , a constant.

3. Exact Solutions

Consider the following Figure 1, which is a section of a cylindrical shell having in-plane axis, α in the direction of the x-axis of shell, β, circumferential in-plane along the y-axis, which is normal to the middle-plane axis z and radius R.

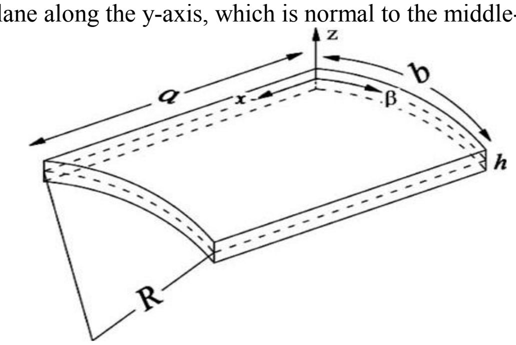

Fig.1. Coordinates of a cylindrical shell (Qatu, 1994)

Let us consider the cylindrical laminated shell as shown in figure 1with length $a/b=1$ under load per unit area, h is the thickness of the shell. If the load is orthogonal to the surface, then Lame' parameters (elastic and shear modulus) of middle surface $A=B=1$ and $1/R_\alpha = 1/R_{\alpha\beta}=0,\ R_\beta$ are substituted in equations (2.5-2.8) to formulate the cylindrical shell equations for TSDTZ. Therefore, the resulting equation (2.4) for the displacement mid surface cylindrical thick shells is rewritten as:

$$\begin{gathered}
\varepsilon_{0\alpha}=\frac{\partial u_0}{\partial\alpha},\quad \varepsilon_{0\beta}=\frac{\partial v_0}{\partial\beta}+\frac{w_0}{R_\beta},\ \varepsilon_{0\alpha\beta}=\frac{\partial v_0}{\partial\alpha},\\
\varepsilon_{0\beta\alpha}=\frac{\partial u_0}{\partial\beta},\ \gamma_{0\alpha z}=\frac{\partial w_0}{\partial\alpha}+\psi_\alpha,\ \ \gamma_{0\beta z}=\frac{\partial w_0}{\partial\beta}-\frac{v_0}{R_\beta}+\psi_\beta,\\
\kappa^{(1)}_\alpha=\frac{\partial\psi_\alpha}{\partial\alpha},\quad \kappa^{(1)}_\beta=\frac{\partial\psi_\beta}{\partial\beta}+\frac{\psi_z}{R_\beta},\ \kappa^{(2)}_\alpha=\frac{\partial\phi_\alpha}{\partial\alpha},\\
\kappa^{(2)}_\beta=\frac{\partial\phi_\beta}{\partial\beta},\ \kappa^{(1)}_{\alpha\beta}=\frac{\partial\psi_\beta}{\partial\alpha},\quad \kappa^{(2)}_{\alpha\beta}=\frac{\partial\phi_\beta}{\partial\alpha}.
\end{gathered} \tag{3.1}$$

Thus, the equations of motion (7) for the cylindrical shell reduces

to

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}(N_\alpha)+\frac{\partial}{\partial\beta}(N_{\beta\alpha})+q_\alpha=(\bar{I}_1u_0+\bar{I}_2\psi_\alpha).\\
&\frac{\partial}{\partial\alpha}(N_{\alpha\beta})+\frac{\partial}{\partial\beta}(N_\beta)+\frac{1}{R_\beta}Q_\beta+q_\beta=(\bar{I}_1v_0+\bar{I}_2\psi_\beta).\\
&\frac{\partial}{\partial\alpha}(Q_\alpha)+\frac{\partial}{\partial\beta}(Q_\beta)-\frac{N_\beta}{R_\beta}+q_n=(\bar{I}_1w_0).\\
&\frac{\partial}{\partial\alpha}(M^{(1)}_\alpha)+\frac{\partial}{\partial\beta}(M^{(1)}_{\beta\alpha})-Q_\alpha+m^{(1)}_\alpha=(\bar{I}_2u_0+\bar{I}_3\psi_\alpha).\\
&\frac{\partial}{\partial\beta}(M^{(1)}_\beta)+\frac{\partial}{\partial\alpha}(M^{(1)}_{\alpha\beta})-Q_\beta+m^{(1)}_\beta=(\bar{I}_2v_0+\bar{I}_3\psi_\beta).\\
&\frac{\partial}{\partial\alpha}(P^{(1)}_\alpha)+\frac{\partial}{\partial\beta}(P^{(1)}_\beta)-(N_z+\frac{M^{(1)}_\beta}{R_\beta})+m_z=(\bar{I}_3\psi_z).\\
&\frac{\partial}{\partial\alpha}(M^{(2)}_\alpha)+\frac{\partial}{\partial\beta}(M^{(2)}_{\beta\alpha})-2P^{(1)}_\alpha+m^{(2)}_\alpha=(\bar{I}_3u_0+\bar{I}_4\varphi_\alpha).\\
&\frac{\partial}{\partial\beta}(M^{(2)}_\beta)+\frac{\partial}{\partial\alpha}(M^{(2)}_{\alpha\beta})-2P^{(1)}_\beta+m^{(2)}_\beta=(\bar{I}_3v_0+\bar{I}_4\varphi_\beta).
\end{aligned} \tag{3.2}$$

Where $\bar{I}_i=\left(I_i+{I_{i+1}}/{R_\beta}\right),\forall\ i=1,2,3,4$.

The above cylindrical shell equation has no exact solution having general boundary conditions. Using the following Fourier series expansion for the midsurface displacements (0th order) field equation (3.3) is used to approximate the solution of the above partial differential equations (3.2):

$$\begin{aligned}
u_0&=\sum_{m,n=1}^{N}u_{0,mn}\,Cos(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\\
v_0&=\sum_{m,n=1}^{N}v_{0,mn}\,Sin(A^*\alpha)\,Cos(B^*\beta)\,e^{-i\omega t},\\
w_0&=\sum_{m,n=1}^{N}w_{0,mn}\,Sin(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\\
\psi_\alpha&=\sum_{m,n=1}^{N}\psi_{\alpha,mn}\,Cos(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t}\\
\psi_\beta&=\sum_{m,n=1}^{N}\psi_{\beta,mn}\,Sin(A^*\alpha)\,Cos(B^*\beta)\,e^{-i\omega t},\\
\psi_z&=\sum_{m,n=1}^{N}\psi_{z,mn}\,Sin(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\\
\phi_\alpha&=\sum_{m,n=1}^{N}\phi_{\alpha,mn}\,Cos(A^*\alpha)\,Sin(B^*\beta)\,e^{-i\omega t},\\
\phi_\beta&=\sum_{m,n=1}^{N}\phi_{\beta,mn}\,Sin(A^*\alpha)\,Cos(B^*\beta)\,e^{-i\omega t}.
\end{aligned} \tag{3.3}$$

Where $A^*=\dfrac{m\pi}{a},B^*=\dfrac{n\pi}{b}$ in which a and b are the dimensions of the mid-shell along the α and β -axes, respectively.

For the free vibration system has no external force on the structure and is freely vibrating. Substituting these equations (3.3) into the Partial differential equations of motion (3.2) yields a set of eight homogenous algebraic systems in terms of its respective components. By collecting the coefficients of the system and by taking the external force vector $\{f\}$ as zero, the resulting equation reduces to an eigenvalue problem. Therefore, the resulting equilibrium equation of motion under free vibration system is written in the following matrix form

$$\{[L]-\lambda[M]\}\{\Delta\}=\{0\}. \tag{3.4}$$

Where $\lambda=\omega^2$, ω is the circular frequency of vibration, $\{\Delta\}$ is the unknown displacement vector. The dynamic shell system stiffness parameters L_{ij} and structure mass matrices M_{ij} are given in appendix C.

4. Numerical Results and Discussion

The above equation (3.4) in the matrix form is solved using MATLAB commercial code. This code is specifically modified for free vibration problems of simply supported cross-ply laminated composite shells. A standard MATLAB routine is used to find the eigenvalues of the matrices. Moreover, the eigenvalues are determined numerically because the stiffness matrix elements contain transcendental functions. The solution is also valid for cylindrical shells having principle radii $R_{\alpha} = R_{\alpha\beta} = \infty$. For the numerical computation we compared the results with the orthotropic material properties of the cylindrical shells having length-to-arc ratio of one unit (i.e. $a/b = 1$), and the Poisson ratio of 0.25. The shear correction factors (K) for both directions are taken as 5/6. The natural frequency, extensional, bending and coupling stiffness parameters for free vibrations are calculated using MATLAB algorithm for laminated composite cylindrical thick shells.

The first five non-dimensional natural frequency parameters (Ω_i) has been calculated for two-ply unsymmetrical [90/0] shells and three-ply symmetric [0/90/0] laminated orthotropic composite cylindrical shells for fixed thickness ratio ($a/h = 10$) and various values of curvature (a/R) ratios by third order shear deformation theory. The results obtained by TSDTZ are then compared with earlier available results first order shear deformation theory by Qatu (FSDTQ), and three dimensional elasticity from finite element method (FEM). This supports us to evaluate the validity of the present TSDTZ theory. Moreover the results obtained by the present method are in good agreement with the existing theory and given in Tables 1-5.

Table 1 and Table 2 show the comparison of present results and those of existing literatures (Asadi et al., 2012; Qatu et al., 2010) for the non-dimensional natural frequency parameters (Ω_i).

TABLE 1

Comparison of first five non-dimensional natural frequency parameters $\Omega_i = \omega_i a^2 \sqrt{\rho E_2/\mathrm{h}^2}$ of 2-ply unsymmetrical $[90/0]$ orthotropic cylindrical shells

$a/b = 1,\ K^2 = 5/6,\ E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ G_{13} = G_{12},\ \nu_{12} = 0.25,\ a/h = 10$

a/R	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2	FSDTQ (Asadi et al., 2012)	13.771	21.037	29.574	31.200	38.073
	TSDTZ (Present)	13.770	21.039	29.587	31.222	37.981
	3D (Asadi et al., 2012; Qatu et al., 2010)	13.772	21.040	29.639	31.411	38.266
1	FSDTQ (Asadi et al., 2012)	10.666	21.705	24.090	30.368	38.722
	TSDTZ (Present)	10.671	21.756	24.117	30.430	38.713
	3D (Asadi et al., 2012; Qatu et al., 2010)	10.686	21.767	24.191	30.614	38.896
0.5	FSDTQ (Asadi et al., 2012)	7.5918	15.284	15.310	20.300	24.073
	TSDTZ (Present)	7.6076	15.322	15.359	20.423	24.157

	3D (Asadi et al., 2012; Qatu et al., 2010)	7.6534	15.437	15.473	20.645	24.364

TABLE 2

Comparison of first five Non-dimensional natural frequency parameters $\Omega_i = \omega_i a^2 \sqrt{\rho E_2/h^2}$ of 3-ply symmetric $[0/90/0]$ orthotropic cylindrical shells

$a/b = 1,\ K^2 = 5/6,\ E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ G_{13} = G_{12},\ \nu_{12} = 0.25,\ a/h = 10$

a/R	Method	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.5	FSDTQ (Asadi et al., 2012)	8.6309	13.581	17.878	20.738	20.976
	TSDTZ (Present)	8.3231	13.387	17.399	20.348	20.894
	3D (Asadi et al., 2012; Qatu et al., 2010)	8.0095	13.192	16.905	19.949	20.811
1	FSDTQ (Asadi et al., 2012)	13.187	18.524	30.564	32.232	34.523
	TSDTZ (Present)	12.891	18.267	30.147	31.224	33.292
	3D (Asadi et al., 2012; Qatu et al., 2010)	12.590	18.005	29.732	30.189	32.037
2	FSDTQ (Asadi et al., 2012)	15.250	17.989	29.491	34.795	34.913
	TSDTZ (Present)	15.040	17.732	29.290	33.665	33.998
	3D (Asadi et al., 2012; Qatu et al., 2010)	14.840	17.468	29.094	32.464	33.046

In order to validate the third order shear deformation theory, the extensional (A_{ij}), bending (D_{ij}) and coupling (B_{ij}) stiffness parameters (Zannon et. al., 2014; Qatu, 1994) are given in Table 3-5 using the Matlab algorithm for laminated composite cylindrical thick shells. It is then compared with the first order shear deformation theory from the literature. There are small discrepancies are seen in the tables 3-5, which is due to the third order shear deformation and the tolerance limitations. Tables 3, 4 and 5 show the extensional stress, coupling, and bending stiffness parameters for [0/90] laminated cylindrical thick shells. While comparing the various stiffness parameters with the existing literature and the present theory, we see that the TSDTZ approximation is more accurate in in comparison with first order shear deformation theory (see Tables1-5 for details).

TABLE 3
Non-dimensional extensional stiffness matrix for $[0/90]$ laminated cylindrical thick shells

$\frac{E_1}{E_2}=15,\ \frac{G_{12}}{E_2}=0.5,\ \frac{G_{13}}{E_2}=0.5,\ \upsilon_{12}=0.3,\ \frac{a}{b}=1,\ \frac{a}{R_\beta}=2,\ \frac{a}{R_{\alpha\beta}}=0,\ \frac{R_\alpha}{R_\beta}=0,\ \frac{a}{h}=10,\ \rho=1.$

(i,j)	Plate Approx. A_{ij}/E_2a^2	FSDTQ Qatu, (1994) $\bar{A}_{ij}/E_2a^2$	$\hat{A}_{ij}/E_2a^2$	TSDTZ (Present) Exact Integration $\bar{A}_{ij}/E_2a^2$	$\hat{A}_{ij}/E_2a^2$	TSDTZ (Present) Third order $\bar{A}_{ij}/E_2a^2$	$\hat{A}_{ij}/E_2a^2$
$(1,1)$	0.804829	0.769634	NA	0.71680	NA	0.71806	NA
$(2,2)$	0.804829	NA	0.772156	NA	0.76962	NA	0.7706
$(6,6)$	0.050000	0.049999	0.050167	0.04750	0.05250	0.04758	0.05258

TABLE 4
Non-dimensional coupling stiffness matrix for $[0/90]$ laminated cylindrical thick shells

$\frac{E_1}{E_2}=15,\ \frac{G_{12}}{E_2}=0.5,\ \frac{G_{13}}{E_2}=0.5,\ \upsilon_{12}=0.3,\ \frac{a}{b}=1,\ \frac{a}{R_\beta}=2,\ \frac{a}{R_{\alpha\beta}}=0,\ \frac{R_\alpha}{R_\beta}=0,\ \frac{a}{h}=10,\ \rho=1.$

(i,j)	Plate Approx. B_{ij}/E_2a^2	FSDTQ Qatu, (1994) $\bar{B}_{ij}/E_2a^2$	$\hat{B}_{ij}/E_2a^2$	TSDTZ (Present) Exact Integration $\bar{B}_{ij}/E_2a^2$	$\hat{B}_{ij}/E_2a^2$	TSDTZ (Present) Third order $\bar{B}_{ij}/E_2a^2$	$\hat{B}_{ij}/E_2a^2$
$(1,1)$	-1.760563	-1.626488	NA	-1.6348	NA	-1.6348	NA
$(2,2)$	1.760563	NA	1.634540	NA	1.6264	NA	1.6264
$(6,6)$	0	0.008329	-0.008379	0.007639	-0.007945	0.008218	-0.008257

TABLE 5
Non-dimensional bending stiffness matrix for $[0/90]$ laminated cylindrical thick shells

$\frac{E_1}{E_2}=15,\ \frac{G_{12}}{E_2}=0.5,\ \frac{G_{13}}{E_2}=0.5,\ \upsilon_{12}=0.3,\ \frac{a}{b}=1,\ \frac{a}{R_\beta}=2,\ \frac{a}{R_{\alpha\beta}}=0,\ \frac{R_\alpha}{R_\beta}=0,\ \frac{a}{h}=10,\ \rho=1.$

(i,j)	Plate Approx. $D_{ij}/E_2 a^2$	FSDTQ Qatu, (1994) $\bar{D}_{ij}/E_2 a^2$	$\hat{D}_{ij}/E_2 a^2$	TSDTZ (Present) Exact Integration $\bar{D}_{ij}/E_2 a^2$	$\hat{D}_{ij}/E_2 a^2$	TSDTZ (Present) Third order $\bar{D}_{ij}/E_2 a^2$	$\hat{D}_{ij}/E_2 a^2$
$(1,1)$	0.670691	0.624178	NA	0.6707	NA	0.67069	NA
$(2,2)$	0.697091	NA	0.630454	NA	0.67069	NA	0.67069
$(6,6)$	0.041667	0.042892	0.041918	0.04165	0.04166	0.04167	0.04167

CONCLUSIONS

This paper presented a third order shear deformation theory for laminated thick shells by Zannon et al. (TSDTZ) and solved for free vibrational characteristics of cylindrical thick shells with simply supported boundary conditions and cross-ply laminates. The present results are compared with the 3D theory of elasticity and first order shear deformation theory available in the literature. The natural frequencies of the first five parameters for the cylindrical shells using TSDTZ are in good agreement with the 3D theory of elasticity. The results obtained here show that the present theory (TSDTZ) is more accurate than the (FSDTQ) when compared to three dimensional theory of elasticity. Also, TSDTZ offers many other advantages in the accurate representations such as extensional , coupling, and stress stiffness parameters, as shown in Tables 3 to 5 and is mainly due to the inclusion of the term $(1+\frac{z}{R})$ in the mathematical formulation of third order shear deformation theory. We anticipate that this theory would be useful for many researchers in the development of complex geometry of shell deformation theory for further future applications.

APPENDIX A

The total shell strain energy U can be written in terms of the midsurface strains and stress resultants and is defined as (Zannon et al., 2014)

$$U = \frac{1}{2}\int_V \{\sigma_\alpha \varepsilon_\alpha + \sigma_\beta \varepsilon_\beta + \sigma_z \varepsilon_z + \sigma_{\alpha\beta}\varepsilon_{\alpha\beta} + \sigma_{\alpha z}\gamma_{\alpha z} + \sigma_{\beta z}\gamma_{\beta z}\} dV$$

$$= \frac{1}{2}\int_\alpha\int_\beta \{N_\alpha \varepsilon_{0\alpha} + N_\beta \varepsilon_{0\beta} + N_z \varepsilon_{0z} + N_{\alpha\beta}\varepsilon_{0\alpha\beta} + N_{\beta\alpha}\varepsilon_{0\beta\alpha} + M_\alpha^{(1)}\kappa_\alpha^{(1)} + M_\alpha^{(2)}\kappa_\alpha^{(2)}$$

$$+ M_\beta^{(1)}\kappa_\beta^{(1)} + M_\beta^{(2)}\kappa_\beta^{(2)} + M_{\alpha\beta}^{(1)}\kappa_{\alpha\beta}^{(1)} + M_{\alpha\beta}^{(2)}\kappa_{\alpha\beta}^{(2)} + M_{\beta\alpha}^{(1)}\kappa_{\beta\alpha}^{(1)} + M_{\beta\alpha}^{(2)}\kappa_{\beta\alpha}^{(2)} + Q_\alpha \gamma_{0\alpha z}$$

$$+ Q_\beta \gamma_{0\beta z} + P_\alpha^{(1)} G^{(1)} + P_\alpha^{(2)} G^{(2)} + P_\beta^{(1)} E^{(1)} + P_\beta^{(2)} E^{(2)}\} AB d\alpha\, d\beta.$$

The total external work W of the thick shell can be written as (Zannon et al., 2014):

$$W = \int_\alpha\int_\beta \{q_\alpha u_0 + q_\beta v_0 + q_n w_0 + m_\alpha^{(1)} G^{(1)} + m_\alpha^{(2)} G^{(2)} + m_z \psi_z$$

$$+ m_\beta^{(1)} E^{(1)} + m_\beta^{(2)} E^{(2)}\} AB\, d\alpha\, d\beta.$$

The kinetic energy T of the thick shell can be written as (Zannon et al., 2014):

$$
\begin{aligned}
T &= \frac{1}{2}\int_V \{u^2 + v^2 + w^2\} dV \\
&= \frac{1}{2}\int_\alpha \int_\beta \ \{(u_0{}^2 + v_0{}^2 + w_0{}^2)\left(I_1 + I_2(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_3}{R_\alpha R_\beta} \right) \\
&+ (\psi_\alpha^2 + \psi_z^2)\left(I_3 + I_4(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_5}{R_\alpha R_\beta} \right) \\
&+ 2(u_0\psi_\alpha + v_0\psi_\beta + w_0\psi_z)\left(I_2 + I_3(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_4}{R_\alpha R_\beta} \right) \\
&+ 2(u_0\varphi_\alpha + v_0\varphi_\beta + \psi_\beta^2)\left(I_4 + I_5(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_6}{R_\alpha R_\beta} \right) \\
&+ 2(\varphi_\alpha\psi_\alpha + \varphi_\beta\psi_\beta)\left(I_5 + I_6(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_7}{R_\alpha R_\beta} \right) \\
&+ (\varphi_\alpha^2 + \varphi_\beta^2)\left(I_7 + I_8(\frac{1}{R_\alpha} + \frac{1}{R_\beta}) + \frac{I_9}{R_\alpha R_\beta} \right)\} AB\, dz\, d\alpha\, d\beta
\end{aligned}
$$

APPENDIX B

$$
\begin{bmatrix} N_\alpha \\ N_\beta \\ N_z \\ N_{\alpha\beta} \\ N_{\beta\alpha} \\ M_\alpha^{(1)} \\ M_\beta^{(1)} \\ M_{\alpha\beta}^{(1)} \\ M_{\beta\alpha}^{(1)} \\ M_\alpha^{(2)} \\ M_\beta^{(2)} \\ M_{\alpha\beta}^{(2)} \\ M_{\beta\alpha}^{(2)} \end{bmatrix}
=
\begin{bmatrix}
\overline{A}_{11} & A_{12} & A_{13} & \overline{A}_{16} & A_{16} & \overline{B_{11}} & B_{12} & \overline{B_{16}} & B_{16} & \overline{D_{11}} & D_{12} & \overline{D_{16}} & D_{16} \\
A_{12} & \hat{A}_{22} & A_{23} & A_{26} & \hat{A}_{26} & B_{12} & \hat{B}_{22} & B_{26} & \hat{B}_{26} & D_{12} & \hat{D}_{22} & D_{26} & \hat{D}_{26} \\
A_{13} & A_{23} & A_{33} & \overline{A_{36}} & A_{36} & B_{13} & B_{23} & \overline{B_{36}} & B_{36} & D_{13} & D_{23} & \overline{D_{36}} & D_{36} \\
\overline{A}_{16} & A_{26} & \overline{A}_{36} & \overline{A}_{66} & A_{66} & \overline{B}_{16} & B_{26} & \overline{B}_{66} & B_{66} & \overline{D}_{16} & D_{26} & \overline{D}_{66} & D_{66} \\
A_{16} & \hat{A}_{26} & A_{36} & A_{66} & \hat{A}_{66} & B_{16} & \hat{B}_{26} & B_{66} & \hat{B}_{66} & D_{16} & \hat{D}_{26} & D_{66} & \hat{D}_{66} \\
\overline{B}_{11} & B_{12} & B_{13} & \overline{B}_{16} & B_{16} & \overline{D}_{11} & D_{12} & \overline{D}_{16} & D_{16} & \overline{E}_{11} & E_{12} & \overline{E}_{16} & E_{16} \\
B_{12} & \hat{B}_{22} & B_{23} & B_{26} & \hat{B}_{26} & D_{12} & \hat{D}_{22} & D_{26} & \hat{D}_{26} & E_{12} & \hat{E}_{22} & E_{26} & \hat{E}_{26} \\
\overline{B}_{16} & B_{26} & \overline{B}_{36} & \overline{B}_{66} & B_{66} & \overline{D}_{16} & D_{26} & \overline{D}_{66} & D_{66} & \overline{E}_{16} & E_{26} & \overline{E}_{66} & E_{66} \\
B_{16} & \hat{B}_{26} & B_{36} & B_{66} & \hat{B}_{66} & D_{16} & \hat{D}_{26} & D_{66} & \hat{D}_{66} & E_{16} & \hat{E}_{26} & E_{66} & \hat{E}_{66} \\
\overline{D}_{11} & D_{12} & D_{13} & \overline{D}_{16} & D_{16} & \overline{E}_{11} & E_{12} & \overline{E}_{16} & E_{16} & \overline{F}_{11} & F_{12} & \overline{F}_{16} & F_{16} \\
D_{12} & \hat{D}_{22} & D_{23} & D_{26} & \hat{D}_{26} & E_{12} & \hat{E}_{22} & E_{26} & \hat{E}_{26} & F_{12} & \hat{F}_{22} & F_{26} & \hat{F}_{26} \\
\overline{D}_{16} & D_{26} & \overline{D}_{36} & \overline{D}_{66} & D_{66} & \overline{E}_{16} & E_{26} & \overline{E}_{66} & E_{66} & \overline{F}_{16} & F_{26} & \overline{F}_{66} & F_{66} \\
D_{16} & \hat{D}_{26} & D_{36} & D_{66} & \hat{D}_{66} & E_{16} & \hat{E}_{26} & E_{66} & \hat{E}_{66} & F_{16} & \hat{F}_{26} & F_{66} & \hat{F}_{66}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{0\alpha} \\ \varepsilon_{0\beta} \\ \varepsilon_{0z} \\ \varepsilon_{0\alpha\beta} \\ \varepsilon_{0\beta\alpha} \\ \kappa_\alpha^{(1)} \\ \kappa_\beta^{(1)} \\ \kappa_{\alpha\beta}^{(1)} \\ \kappa_{\beta\alpha}^{(1)} \\ \kappa_\alpha^{(2)} \\ \kappa_\beta^{(2)} \\ \kappa_{\alpha\beta}^{(2)} \\ \kappa_{\beta\alpha}^{(2)} \end{bmatrix}
$$

$$
\begin{bmatrix} Q_\alpha \\ Q_\beta \\ P_\alpha^{(1)} \\ P_\beta^{(1)} \\ P_\alpha^{(2)} \\ P_\beta^{(2)} \end{bmatrix}
=
\begin{bmatrix}
\overline{A}_{55} & A_{45} & \overline{B}_{55} & B_{45} & \overline{D}_{55} & D_{45} \\
A_{45} & \hat{A}_{44} & B_{45} & \hat{B}_{44} & D_{45} & \hat{D}_{44} \\
\overline{B}_{55} & B_{45} & \overline{D}_{55} & D_{45} & \overline{E}_{55} & E_{45} \\
B_{45} & \hat{B}_{44} & D_{45} & \hat{D}_{44} & E_{45} & \hat{E}_{44} \\
\overline{D}_{55} & D_{45} & \overline{E}_{55} & E_{45} & \overline{F}_{55} & F_{45} \\
D_{45} & \hat{D}_{44} & E_{45} & \hat{E}_{44} & F_{45} & \hat{F}_{44}
\end{bmatrix}
\begin{bmatrix} \gamma_{0\alpha z} \\ \gamma_{0\beta z} \\ G^{(1)} \\ E^{(1)} \\ G^{(2)} \\ E^{(2)} \end{bmatrix}
$$

Where $A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, \overline{A}_{ij}, \overline{B}_{ij}, \overline{D}_{ij}, \overline{E}_{ij}, \overline{F}_{ij}, \hat{A}_{ij}, \hat{B}_{ij}, \hat{D}_{ij}, \hat{E}_{ij}$ and $\hat{F}_{ij}$ are given in Zannon et al., (2014)

APPENDIX C

$$L_{11} = -\overline{A}_{11} \cdot A^{*2} - \hat{A}_{66} \cdot B^{*2}, \; L_{12} = -A_{12} \cdot A^* \cdot B^* - A_{66} \cdot A^* \cdot B^*$$

$$L_{13} = \frac{A_{12} \cdot A^*}{R_\beta}, \; L_{14} = -\overline{B}_{11} \cdot A^{*2} - \hat{B}_{66} \cdot B^{*2}$$

$$L_{15} = -B_{12} \cdot A^* \cdot B^* - B_{66} \cdot A^* \cdot B^*, \; L_{16} = A_{13} \cdot A^* + \frac{B_{12} \cdot A^*}{R_\beta}$$

$$L_{17} = -\overline{D}_{11} \cdot A^{*2} - \hat{D}_{66} \cdot B^{*2}, \; L_{18} = -D_{12} \cdot A^* \cdot B^* - D_{66} \cdot A^* \cdot B^*$$

$$L_{21} = -A_{12} \cdot A^* \cdot B^* - A_{66} \cdot A^* \cdot B^*, \; L_{22} = -\hat{A}_{22} \cdot B^{*2} - \overline{A}_{66} \cdot A^{*2} + \frac{\hat{A}_{44}}{(R_\beta)^2}$$

$$L_{23} = \frac{\hat{A}_{44} \cdot B^* + \hat{A}_{22} \cdot B^*}{R_\beta}, \; L_{24} = -B_{12} \cdot A^* \cdot B^* - B_{66} \cdot A^* \cdot B^*$$

$$L_{25} = -\hat{B}_{22} \cdot B^{*2} - \overline{B}_{66} \cdot A^{*2} + \frac{\hat{A}_{44}}{R_\beta}, \; L_{26} = A_{23} \cdot B^* + \frac{\hat{B}_{22} \cdot B^*}{R_\beta} + \frac{\hat{B}_{44} \cdot B^*}{R_\beta}$$

$$L_{27} = -D_{12} \cdot A^* \cdot B^* - D_{66} \cdot A^* \cdot B^*, \; L_{28} = -\hat{D}_{22} \cdot B^{*2} - \overline{D}_{66} \cdot A^{*2} + \frac{2 \cdot \hat{B}_{44}}{R_\beta} + \frac{\hat{D}_{44}}{(R_\beta)^2}$$

$$L_{31} = \frac{A_{12} \cdot A^*}{R_\beta}, \; L_{32} = \frac{\hat{A}_{44} \cdot B^*}{R_\beta}, \; L_{33} = -\overline{A}_{55} \cdot A^{*2} - \hat{A}_{44} \cdot B^{*2}$$

$$L_{34} = -\overline{A}_{55} \cdot A^* + \frac{B_{12} \cdot A^*}{R_\beta}, \; L_{35} = -\hat{A}_{44} \cdot B^* + \frac{\hat{B}_{22} \cdot B^*}{R_\beta}$$

$$L_{36} = -A^{*2} \cdot \overline{B}_{55} - B^{*2} \hat{B}_{44} - \frac{\hat{A}_{22}}{R_\beta} + \frac{\hat{B}_{22}}{(R_\beta)^2}$$

$$L_{37} = 2 \cdot A^* \cdot \overline{B}_{55} + \frac{D_{12} \cdot A^*}{R_\beta}, \; L_{38} = -2 \cdot B^* \cdot \hat{B}_{44} + \frac{\hat{D}_{22} \cdot B^*}{R_\beta}$$

$$L_{41} = L_{14}, \; L_{42} = L_{24}, \; L_{43} = L_{34}, \; L_{44} = -\overline{D}_{11} \cdot A^{*2} - \hat{D}_{66} \cdot B^{*2} - \overline{A}_{55},$$

$$L_{45} = -D_{12} \cdot A^* \cdot B^* - D_{66} \cdot A^* \cdot B^*, \; L_{46} = B_{13} \cdot A^* + \frac{D_{12} \cdot A^*}{R_\beta} - \overline{B}_{55} \cdot A^*,$$

$$L_{47} = -\overline{E}_{11} \cdot A^* - \hat{E}_{66} \cdot B^{*2} - 2 \cdot \overline{B}_{55}, \; L_{48} = -E_{12} \cdot A^* \cdot B^* - E_{66} \cdot A^* \cdot B^*,$$

$$L_{51} = L_{15}, \; L_{52} = L_{25}, \; L_{53} = L_{35}, \; L_{54} = L_{45}, \; L_{55} = -\hat{D}_{22} \cdot B^* - D_{66} \cdot A^{*2} - \hat{A}_{44},$$

$$L_{56} = \frac{\hat{D}_{22} \cdot B^*}{R_\beta} - \hat{B}_{44} \cdot B^*, \; L_{57} = -E_{12} \cdot A^* \cdot B^* - E_{66} \cdot A^* \cdot B^*,$$

$$L_{58} = -\hat{E}_{22} \cdot B^{*2} - \overline{E}_{66} \cdot A^{*2} - 2 \cdot \hat{B}_{44} - \frac{\hat{D}_{44}}{R_\beta},$$

$$L_{61} = L_{16}, \; L_{62} = L_{26}, \; L_{63} = L_{36}, \; L_{64} = L_{46}, \; L_{65} = L_{56}$$

$$L_{66} = -\overline{D}_{55} \cdot A^{*2} - \hat{D}_{44} \cdot B^{*2} - A_{33} - \frac{B_{23}}{R_\beta} - \frac{\hat{D}_{22}}{(R_\beta)^2}$$

$$L_{67} = -2 \cdot \overline{D}_{55} \cdot A^* + \frac{E_{12} \cdot A^*}{R_\beta},$$

$$L_{68} = -2 \cdot \hat{D}_{55} \cdot B^* + \frac{\hat{E}_{44} \cdot B^*}{R_\beta} + \frac{\hat{E}_{22} \cdot B^*}{R_\beta}, \; L_{71} = L_{17}, \; L_{72} = L_{27},$$

$$L_{73} = L_{37}, \; L_{74} = L_{47}, \; L_{75} = L_{57}, \; L_{76} = L_{67}, \; L_{81} = L_{18}, L_{84} = L_{48}, \; L_{85} = L_{58}$$

$$L_{77} = -\overline{F}_{11} \cdot A^{*2} - \hat{F}_{66} \cdot B^{*2} - 4 \cdot \overline{D}_{55},$$

$$L_{78} = -F_{12} \cdot A^* \cdot B^* - F_{66} \cdot A^* \cdot B^*, \; L_{82} = L_{28}, \; L_{83} = L_{38},, \; L_{86} = L_{68}$$

$$L_{87} = L_{78}, \; L_{88} = -\hat{F}_{22} \cdot B^{*2} - \overline{F}_{66} \cdot A^{*2} - \frac{\hat{E}_{44}}{R_\beta} - \frac{\hat{F}_{44}}{(R_\beta)^2} - 4 \cdot \hat{D}_{44}$$

The mass parameters M_{ij} for the shell is given by Zannon et al., (2014)

$$M_{11} = \overline{I}_1 \ , \ M_{14} = \overline{I}_2 \ , \ M_{12} = M_{13} = M_{15} = M_{16} = M_{17} = M_{18} = 0,$$

$$M_{22} = \overline{I}_1 \ , \ M_{25} = \overline{I}_2 \ , \ M_{21} = M_{23} = M_{24} = M_{26} = M_{27} = M_{28} = 0,$$

$$M_{33} = \overline{I}_1 \ , \ M_{31} = M_{32} = M_{34} = M_{35} = M_{36} = M_{37} = M_{38} = 0,$$

$$M_{41} = \overline{I}_2 \ , M_{44} = \overline{I}_3 \ , \ M_{42} = M_{43} = M_{45} = M_{46} = M_{47} = M_{48} = 0,$$

$$M_{52} = \overline{I}_2 \ , M_{55} = \overline{I}_3 \ , \ M_{51} = M_{53} = M_{54} = M_{56} = M_{57} = M_{58} = 0,$$

$$M_{66} = \overline{I}_3 \ , \ M_{61} = M_{62} = M_{63} = M_{64} = M_{65} = M_{67} = M_{68} = 0,$$

$$M_{77} = \overline{I}_4 \ , M_{71} = M_{72} = M_{73} = M_{74} = M_{75} = M_{76} = M_{78} = 0,$$

$$M_{88} = \overline{I}_4 \ , M_{82} = M_{81} = M_{83} = M_{84} = M_{85} = M_{86} = M_{87} = 0.$$

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