On (α, β)-Fuzzy H_v-Submodules of H_v-Modules

Arvind Kumar Sinha, Manoj Kumar Dewangan

Abstract— **In this paper we introduce the concept of an (α, β) -fuzzy H_v -submodule of an H_v -module by using the notion of "belongingness $(\in)$ " and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. Since the concept of $(\in, \in \vee q)$ -fuzzy H_v -submodules is an important and useful generalization of ordinary fuzzy H_v -submodules, we discuss some fundamental aspects of $(\in, \in \vee q)$ -fuzzy H_v-submodules. Also we extend the concept of a fuzzy H_v -ideal with thresholds to the concept of a fuzzy H_v -submodule with thresholds Mathematics Subject Classification 20N20**

Index Terms— **Hyperstructure, H_v-module, Fuzzy set, Fuzzy H_v-submodule.**

INTRODUCTION

The algebraic hyperstructure is a natural generalization of the usual algebraic structures which was first initiated by Marty [12]. After the pioneering work of F. Marty, algebraic hyperstructures have been developed by many researchers. A short review of which appears in [14]. A recent book on hyperstructures [15] points out their applications in geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Vougiouklis [22] introduced a new class of hyperstructures so-called H_v-structure, and Davvaz [3] surveyed the theory of H_v-structures. The H_v-structures are hyperstructures where equality is replaced by non-empty intersection.

The concept of fuzzy sets was first introduced by Zadeh [13] and then fuzzy sets have been used in the reconsideration of classical mathematics. In particular, the notion of fuzzy subgroup was defined by Rosenfeld [1] and its structure was thereby investigated. Liu [23] introduced the notions of fuzzy subrings and ideals. Using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of (α, β)-fuzzy subgroup where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced in [17]. The most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in \vee q)$-fuzzy subgroups, the detailed study of which may be found in [19]. The concept of an $(\in, \in \vee q)$-fuzzy subring and ideal of a ring have been introduced in [18] and the concept of $(\in, \in \vee q)$-fuzzy subnear-ring and ideal of a near-ring have been introduced in [2]. Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now studied both from the theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by many authors. In [4, 5, 7], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy H_v -subgroups, fuzzy H_v -ideals and fuzzy H_v-submodules, which are generalizations of the concepts of Rosenfeld's fuzzy subgroups, fuzzy ideals and fuzzy submodules. The concept of a fuzzy H_v -ideal and H_v-subring has been studied further in [6, 8]. Davvaz [9] introduced the notion of (α, β) -fuzzy H_v -ideals of H_v-rings. This paper continues this line of research for (α, β)-fuzzy H_v-submodule of an H_v-module.

The paper is organized as follows: In Section 2, we recall some basic definitions and results about H_v-structures. In Section 3, we introduce the concept of (α, β) -fuzzy H_v-submodule of an H_v-module and investigate related results. Since the concept of $(\in, \in \vee q)$ -fuzzy H_v-submodule is an important and useful generalization of ordinary fuzzy H_v-submodules, some fundamental aspects of $(\in, \in \vee q)$-fuzzy H_v-submodules have been discussed in Section 4. Also we extend the concept of a fuzzy H_v-ideal with thresholds to the concept of a fuzzy H_v-submodule with thresholds.

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Arvind Kumar Sinha, Department of Mathematics, NIT Raipur, Chhattisgarh, India
Manoj Kumar Dewangan, Department of Mathematics, NIT Raipur, Chhattisgarh, India

Basic Definitions

We first give some basic definitions for proving the further results.

Definition 2.1. [11] Let X be a non-empty set. A mapping $\mu: X \to [0,1]$ is called a fuzzy set in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x) \quad \forall x \in X.$

Definition 2.2. [20] Let G be a non–empty set and $*: G \times G \to \wp^*(G)$ be a hyperoperation, where $\wp^*(G)$ is the set of all the non-empty subsets of G. Where $A * B = \bigcup_{a\in A,\, b\in B} a * b,\ \forall A, B \subseteq G$.

The $*$ is called weak commutative if $x * y \cap y * x \neq \phi, \quad \forall x, y \in G$.
The $*$ is called weak associative if $(x * y) * z \cap x * (y * z) \neq \phi, \quad \forall x, y, z \in G$.

A hyperstructure $(G, *)$ is called an H_v–group if

(i) $*$ is weak associative.
(ii) $a * G = G * a = G, \quad \forall a \in G$ (Reproduction axiom).

Definition 2.3. [20] An H_v–ring is a system $(R, +, \cdot)$ with two hyperoperations satisfying the ring-like axioms:

(i) $(R, +)$ is an H_v-group, that is,

$((x + y) + z) \cap (x + (y + z)) \neq \phi \quad \forall x, y \in R,$

$a + R = R + a = R \quad \forall a \in R;$

(ii) $(R, \cdot)$ is an H_v-semigroup;

(iii) $(\cdot)$ is weak distributive with respect to $(+)$, that is, for all $x, y, z \in R$ $\begin{array}{l}(x\cdot(y+z))\cap(x\cdot y + x\cdot z)\neq\phi,\\ ((x+y)\cdot z)\cap(x\cdot z + y\cdot z)\neq\phi.\end{array}$

Definition 2.4. [10] Let R be an H_v-ring. A nonempty subset I of R is called a left (resp., right) H_v-ideal if the following axioms hold:

(i) $(I, +)$ is an H_v-subgroup of $(R, +)$,
(ii) $R \cdot I \subseteq I$ (resp., $I \cdot R \subseteq I$).

Definition 2.5. [10] Let $(R, +, \cdot)$ be an H_v–ring and μ a fuzzy subset of R. Then μ is said to be a left (resp., right) fuzzy H_v-ideal of R if the following axioms hold:

(1) $\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\} \forall x, y \in R,$
(2) For all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y),$
(3) For all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and $\min\{\mu(a), \mu(x)\} \leq \mu(z),$ (4) $\mu(y) \leq \inf\{\mu(z) : z \in x \cdot y\}$ respectively $\mu(x) \leq \inf\{\mu(z) : z \in x \cdot y\} \quad \forall x, y \in R.$

Definition 2.6. [20] A nonempty set M is called an H_v–module over an H_v–ring R if (M, +) is a weak commutative H_v–group and there exists a map $.: R \times M \to \wp^*(M), (r, x) \to r.x$ Such that for all $a, b \in R$ and $x, y \in M$, we have

$$\begin{array}{l}(a.(x+y))\cap(a.x+a.y)\neq\phi,\\ ((x+y).a)\cap(x.a+y.a)\neq\phi,\\ (a.(b.x))\cap((a.b).x)\neq\phi.\end{array}$$

Note that by using fuzzy sets, we can consider the structure of H_v-module on any ordinary module which is a generalization of a module.

Definition 2.7. [20] A fuzzy set μ in M is called a fuzzy H_v –submodule of M if (1) $\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\} \forall x, y \in M,$

(2) For all $x, a \in M$ there exists $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and $\min\{\mu(a), \mu(x)\} \leq \inf\{\mu(y), \mu(z)\},$

(3) $\mu(y) \leq \inf\{\mu(z) : z \in x \cdot y\}$ for all $y \in M$ and $x \in R.$

Definition 2.8. [20] Let μ be a fuzzy subset of R. If there exist a $t \in (0, 1]$ and an $x \in R$ such that

$$\mu(y)=\begin{cases} t & if\ y=x \\ 0 & if\ y\neq x \end{cases}$$

Then μ is called a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.9. [20] Let μ be a fuzzy subset of R and x_t be a fuzzy point.

(1) If $\mu(x)\geq t$, then we say x_t belongs to μ, and write $x_t \in \mu$.

(2) If $\mu(x)+t>1$, then we say x_t is quasi-coincident with μ, and write $x_t q\mu$. $(3) x_t \in \vee q\mu \Leftrightarrow x_t \in \mu \ or \ x_t q\mu.$

$(4) x_t \in \wedge q\mu \Leftrightarrow x_t \in \mu \ and \ x_t q\mu.$

In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in\vee q$ or $\in\wedge q$ with $\alpha\neq\in\wedge q$, which was introduced by Bhakat and Das [9].

Definition 2.10. [20] Let R be an H_v–ring. A fuzzy subset μ of R is said to be an (α,β)–fuzzy left (right) H_v–ideals of R if for all $t,r\in(0,1]$,

$(1) x_t\alpha\mu, y_r\alpha\mu \Rightarrow z_{t\wedge r}\beta\mu \quad \forall z\in x+y,$

$(2) x_t\alpha\mu, a_r\alpha\mu \Rightarrow y_{t\wedge r}\beta\mu$ for some $y\in R$ with $x\in a+y,$

$(3) x_t\alpha\mu, a_r\alpha\mu \Rightarrow z_{t\wedge r}\beta\mu$ for some $z\in R$ with $x\in z+a,$

$(4) y_t\alpha\mu, x\in R \Rightarrow z_t\beta\mu \quad \forall z\in x.y \quad (x_t\alpha\mu, y\in R \Rightarrow z_t\beta\mu \quad \forall z\in x.y).$

3. (α,β)–Fuzzy H_v-Submodules

In what follows, let M denote an H_v-module over an H_v–ring R unless otherwise specified. We start by defining the notion of (α,β)–fuzzy H_v–submodules.

Definition 3.1. A fuzzy set μ in M is called a (α,β)-fuzzy left (right) H_v –submodule of M if for all $t,r\in(0,1]$,

$(1) \forall x,y\in M, x_t, y_r\alpha\mu \Rightarrow z_{t\wedge r}\beta\mu \quad \forall z\in x+y,$

$(2) \forall x,a\in M, x_t, a_r\alpha\mu \Rightarrow (y\wedge z)_{t\wedge r}\beta\mu$ for some $y,z\in M$ with $x\in(a+y)\cap(z+a),$

$(3) \forall y\in M, x\in R, y_t\alpha\mu \Rightarrow z_t\beta\mu \quad \forall z\in x.y \quad (\forall y\in M, x\in R, y_t\alpha\mu \Rightarrow z_t\beta\mu \quad \forall z\in y.x).$

In this paper we present all the proofs for left H_v–submodules. Similar results hold for right H_v–submodules.

Proposition 3.2. Let M be an H_v –module. Every $(\in\vee q,\in\vee q)$ -fuzzy left (right) H_v –submodule of M is an $(\in,\in\vee q)$–fuzzy left (right) H_v–submodule of M.

Proof. Let μ be an $(\in\vee q,\in\vee q)$–fuzzy left H_v–submodule of M.

(i) Suppose that $x,y\in M$ and $t,r\in[0,1]$ be such that $x_t, y_r\in\mu.$ Then $x_t, y_r\in\vee q\mu,$ and so $z_{t\wedge r}\in\vee q\mu,$ for all $z\in x+y.$

(ii) Now let $x,a\in M$ and $t,r\in(0,1]$ be such that $x_t, a_r\in\mu.$ Then $x_t, a_r\in\vee q\mu$ which implies $(y\wedge z)_{t\wedge r}\in\vee q\mu$ for some $y,z\in M$ with $x\in(a+y)\cap(z+a).$

(iii) Let $x,y\in M$ and $t\in(0,1]$ be such that $y_t\in\mu.$ Then $y_t\in\vee q\mu$ which implies $z_t\in\vee q\mu,$ for all $z\in x.y.$

Proposition 3.3. Let M be an H_v–module. Every $(\in,\in)$-fuzzy left (right) H_v–submodule of M is an $(\in,\in\vee q)$–fuzzy left (right) H_v–submodule of M.

Lemma 3.4. If μ is a fuzzy left (right) H_v-submodules of M, then the characteristic function χ_A of μ is an $(\in,\in)$-fuzzy left (right) H_v-submodule of M.

Now, we give the main result on general (α,β)-fuzzy left (right) H_v-submodules of H_v-modules.

Theorem 3.5. Let μ be a non-zero (α,β) –fuzzy left (right) H_v -submodule of M. Then the set $Supp\mu = \{x \in M : \mu(x) > 0\}$ is a left (right) H_v-submodule of M.

A fuzzy subset μ of an H_v-module M is said to be proper if $\operatorname{Im}\mu$ has at least two elements. Two fuzzy subsets are said to be equivalent if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

Theorem 3.6. Let M have proper H_v-submodules. A proper $(\in,\in)$-fuzzy H_v-submodule μ of M such that card $\operatorname{Im}\mu \geq 3$, can be expressed as the union of two proper non-equivalent $(\in,\in)$-fuzzy H_v-submodules of M.

4. $(\in,\in\vee q)$-fuzzy H_v-submodules

In this section, we consider a special case of (α,β)-fuzzy H_v-submodules. An $(\in,\in\vee q)$-fuzzy H_v-submodule is an important and useful generalization of ordinary fuzzy H_v-submodule.

Definition 4.1. Let M be an H_v-module. A fuzzy subset μ of M is said to be an $(\in,\in\vee q)$-fuzzy left (right) H_v-submodule of M if for all $t,r \in (0,1]$

$(i)\, x_t \in \mu, y_r \in \mu \Rightarrow z_{t\wedge r} \in \vee q\mu, \quad \forall z \in x+y;$

$(ii)\, x_t \in \mu, a_r \in \mu \Rightarrow (y\wedge z)_{t\wedge r} \in \vee q\mu,$ for some $y,z \in M$ with $x \in (a+y)\cap(z+a);$ $(iii)\, y_t \in \mu, x \in M \Rightarrow z_t \in \vee q\mu, \quad \forall z \in x.y \left(x_t \in \mu, y \in M \Rightarrow z_t \in \vee q\mu, \quad \forall z \in x.y\right).$

It is easy to see that for any subset μ of M, χ_A is an -fuzzy left (right) H_v-submodule of M if and only if μ is a left (right) H_v-submodule of M.

Proposition 4.2. Conditions (i)-(iii) in Definition 4.1, are respectively equivalent to the following:

$(1)\, \mu(x)\wedge\mu(y)\wedge 0.5 \underset{z\in x+y}{\wedge} \mu(z), \forall x,y \in M;$

$(2)\, \forall x,a \in M$ there exists $y,z \in M$ such that $x \in (a+y)\cap(z+a)$ and $\mu(a)\wedge\mu(x)\wedge 0.5 \leq \mu(y)\wedge\mu(z);$

$(3)\, \mu(y)\wedge 0.5 \underset{z\in x.y}{\wedge} \mu(z), \forall x,y \in M \left(\mu(x)\wedge 0.5 \underset{z\in x.y}{\wedge} \mu(z), \forall x,y \in M\right).$

Proof. $(i \Rightarrow 1)$: Suppose that $x,y \in M$. We consider the following cases:

$(a)\, \mu(x)\wedge\mu(y) < 0.5,$

$(b)\, \mu(x)\wedge\mu(y) \geq 0.5.$

Case a: Assume that there exists $z \in x+y$ such that $\mu(z) < \mu(x)\wedge\mu(y)\wedge 0.5,$ which implies $\mu(z) < \mu(x)\wedge\mu(y).$ Choose t such that $\mu(z) < t < \mu(x)\wedge\mu(y).$ Then $x_t, y_t \in \mu$ but $z_t \overline{\in\vee q\mu}$ which contradicts (i).

Case b: Assume that $\mu(z) < 0.5$ for some $z \in x+y.$ Then $x_{0.5}, y_{0.5} \in \mu,$ but $z_{0.5}\overline{\in\vee q\mu},$ a contradiction.

Hence (1) holds.

$(ii \Rightarrow 2)$: Suppose that $x,a \in M.$ We consider the following cases:

$(a)\, \mu(x)\wedge\mu(a) < 0.5,$

$(b)\, \mu(x)\wedge\mu(a) \geq 0.5.$

Case a: Assume that for all $y,z \in M$ with $x \in (a+y)\cap(z+a),$ we have $\mu(y)\wedge\mu(z) < \mu(x)\wedge\mu(a).$

Choose t such that $\mu(y)\wedge\mu(z) < t < \mu(x)\wedge\mu(a)$ and $t+\mu(y)\wedge\mu(z) < 1.$ Then $x_t, a_t \in \mu,$ but $(y\wedge z)_t \overline{\in\vee q\mu},$ which contradicts (ii).

Case b: Assume that for all $y,z \in M$ with $x \in (a+y)\cap(z+a),$ we have $\mu(y)\wedge\mu(z) < \mu(x)\wedge\mu(a)\wedge 0.5.$

Then $x_{0.5}, a_{0.5} \in \mu,$ but $(y\wedge z)_{0.5} \overline{\in\vee q\mu},$ which contradicts (ii).

Hence (2) holds.

$(iii \Rightarrow 3)$: Suppose $x, y \in M$. We consider the following cases:

$(a)\, \mu(y) < 0.5,$

$(b)\, \mu(y) \geq 0.5.$

Case a: Assume that there exists $z \in x.y$ such that $\mu(z) < \mu(y) \wedge 0.5$, which implies $\mu(z) < \mu(y)$. Choose t such that $\mu(z) < t < \mu(y)$. Then $y_t \in \mu$, but $z_t \overline{\in \vee q} \mu$, which contradicts (iv).

Case b: Assume that $\mu(z) < 0.5$ for some $z \in x.y$ Then $y_{0.5} \in \mu$, but $z_{0.5} \overline{\in \vee q \mu}$, a contradiction. Hence (3) holds.

$(1 \Rightarrow i)$: Let $x_t, y_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. For every $z \in x + y$ we have $\mu(z) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge r \wedge 0.5$. If $t \wedge r > 0.5$, then $\mu(z) \geq 0.5$ which implies $\mu(z) + t \wedge r > 1$. If $t \vee r \leq 0.5$, then $\mu(z) \geq t \wedge r$.

Therefore $z_{t \wedge r} \in \vee q \mu$ for all $z \in x + y$.

$(2 \Rightarrow ii)$: Let $x_t, a_r \in \mu$. Then $\mu(x) \geq t$ and $\mu(a) \geq r$. Now, for some $y, z \in M$ with $x \in (a + y) \cap (z + a)$, we have

$\mu(y) \wedge \mu(z) \geq \mu(a) \wedge \mu(x) \wedge 0.5 \geq t \wedge r \wedge 0.5$. If $t \wedge r > 0.5$, then $\mu(y) \wedge \mu(z) \geq 0.5$ which implies $\mu(y) \wedge \mu(z) + t \wedge r > 1$.

If $t \vee r \leq 0.5$, then $\mu(y) \wedge \mu(z) \geq t \wedge r$.

Therefore $(y \wedge z)_{t \wedge r} \in \vee q \mu$ Hence (ii) holds.

$(3 \Rightarrow iii)$: Let $y_t \in \mu$ and $x \in M$. Then $\mu(y) \geq t$. For every $z \in x.y$ we have $\mu(z) \geq \mu(y) \wedge 0.5 \geq t \wedge 0.5$.

If $t > 0.5$, then $\mu(z) \geq 0.5$ which implies $\mu(z) + t > 1$.

If $t \leq 0.5$, then $\mu(z) \geq t$.

Therefore $z_t \in \vee q \mu$ for all $z \in x.y$.

By Definition 4.1 and Proposition 4.2, we immediately get:

Corollary 4.3. A fuzzy subset μ of an H_v-module M is an $(\in, \in \vee q)$-fuzzy left (right) H_v-module of M if and only if the conditions (1)-(3) in Proposition 4.2 hold.

Now, we characterize $(\in, \in \vee q)$-fuzzy left (right) H_v-modules by their level H_v-modules.

Theorem 4.4. Let M be an H_v-module and μ a fuzzy subset of M. If μ is an $(\in, \in \vee q)$-fuzzy left (right) H_v-submodule of M, then for all $0 < t \leq 0.5$, μ_t is an empty set or a left (right) H_v-submodule of M. Conversely, if $\mu_t (\neq \phi)$ is a left (right) H_v-submodule of M for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee q)$-fuzzy left (right) H_v-module of M.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy left H_v-submodule of M and $0 < t \leq 0.5$. Let $x, y \in \mu_t$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Now $\bigwedge_{z \in x+y} \mu(z) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t$. Therefore for every $z \in x + y$ we have $\mu(z) \geq t$ or $z \in \mu_t$, so $x + y \subseteq \mu_t$. Hence for every $a \in \mu_t$ we have $a + \mu_t \subseteq \mu_t$. Now, let $x, a \in \mu_t$. Then there exists $y \in M$ such that $x \in a + y$ and $\mu(a) \wedge \mu(x) \wedge 0.5 \leq \mu(y)$. From $x, a \in \mu_t$, we have $\mu(x) \geq t$ and $\mu(a) \geq t$, and so $t = t \wedge t \wedge 0.5 \leq \mu(a) \wedge \mu(x) \wedge 0.5 \leq \mu(y)$. Hence $y \in \mu_t$, and this proves that $\mu_t \subseteq a + \mu_t$. Now, let $y \in \mu_t$, and $x \in M$. Then $\mu(y) \geq t$ and so $\bigwedge_{z \in x.y} \mu(z) \geq \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t$. Therefore for every $z \in x.y$ we have $\mu(z) \geq t$ or $z \in \mu_t$, so $x.y \subseteq \mu_t$.

Conversely, let μ be a fuzzy subset of M such that $\mu_t (\neq \phi)$ is a left H_v-submodule of M for all $0 < t \leq 0.5$. For every $x, y \in M$, we can write

$\mu(x) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = t_0,$

$\mu(y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = t_0,$

Then $x \in \mu_{t_0}$ and $y \in \mu_{t_0}$, so $x + y \subseteq \mu_{t_0}$. Therefore for every $z \in x + y$ we have $\mu(z) \geq t_0$ which implies $\underset{z\in x+y}{\wedge} \mu(z) \geq t_0$, and hence condition (1) of Proposition 4.2 is verified.

To verify the second condition, for every $a, x \in M$, we put $t_1 = \mu(a) \wedge \mu(x) \wedge 0.5$. Then $x \in \mu_{t_1}$ and $a \in \mu_{t_1}$. So there exists $y, z \in \mu_{t_1}$ such that $x \in (a + y) \cap (z + a)$. Since $y, z \in \mu_{t_1}$ we have $\mu(y) \geq t_1$ and $\mu(z) \geq t_1$ or $\mu(y) \wedge \mu(z) \geq \mu(a) \wedge \mu(x) \wedge 0.5$.

Now, let $x, y \in M$. We can write $\mu(y) \geq \mu(y) \wedge 0.5 = t_0$. Then $y \in \mu_{t_0}$ and so $x.y \subseteq \mu_{t_0}$. Therefore for every $z \in x.y$ we have $\mu(z) \geq t_0$ which implies $\underset{z\in x.y}{\wedge} \mu(z) \geq t_0$, and hence condition (3) of Proposition 4.2 is verified.

Naturally, a corresponding result is true when μ_t is a left H_v-submodule of M for all $t \in (0.5, 1]$.

Theorem 4.5. Let M be an H_v-module and μ a fuzzy subset of M. Then $\mu_t (\neq \phi)$ is a left (right) H_v-submodule of M for all $t \in (0.5, 1]$ if and only if

$(1)\, \mu(x) \wedge \mu(y) \leq \underset{z\in x+y}{\wedge} (\mu(z) \vee 0.5)$, for all $x, y \in M$;

(2) for all $x, a \in M$ there exists $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and $\mu(a) \wedge \mu(x) \leq \mu(y) \vee \mu(z) \vee 0.5;$

$(3)\, \mu(y) \leq \underset{z\in x.y}{\wedge} (\mu(z) \vee 0.5)$, for all $x, y \in M$.

Proof. $(\Rightarrow)$: If there exist $x, y, z \in M$ with $z \in x + y$ such that $\mu(z) \vee 0.5 < \mu(x) \wedge \mu(y) = t$, then $t \in (0.5, 1]$, $\mu(z) < t$, $x \in \mu_t$, and $y \in \mu_t$. Since $x, y \in \mu_t$ and μ_t is a left H_v-submodule so $x + y \subseteq \mu_t$ and $\mu(z) \geq t$, for all $z \in x + y$, which is in contradiction with $\mu(z) < t$. Therefore $\mu(x) \wedge \mu(y) \geq \mu(z) \vee 0.5$, for all $x, y, z \in M$ with $z \in x + y$, which implies $\mu(x) \wedge \mu(y) \geq \underset{z\in x+y}{\wedge} (\mu(z) \vee 0.5)$, for all $x, y \in M$. Hence (1) holds.

Now, assume that there exist $x_0, a_0 \in M$ such that for all $y, z \in M$ with $x_0 \in (a_0 + y) \cap (z + a_0)$, the following inequality holds: $\mu(y) \vee 0.5 < \mu(a_0) \wedge \mu(x_0) = t$. Then $t \in (0.5, 1], x_0 \in \mu_t, a_0 \in \mu_t$ and $\mu(y) < t, \mu(z) < t$. Since $x_0, a_0 \in \mu_t$ and μ_t is a left H_v-submodule , there exists $y_0, z_0 \in \mu_t$ such that $x_0 \in (a_0 + y_0) \cap (z_0 + a_0)$. From $y_0, z_0 \in \mu_t$, we get $\mu(y_0) \geq t, \mu(z_0) \geq t$, which is in contradiction with $\mu(y_0) < t, \mu(z_0) < t$. Therefore for all $x, a \in M$ there exists $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and $\mu(a) \wedge \mu(x) \leq \mu(y) \vee \mu(z) \vee 0.5$. Hence (2) holds.

Now, if there exist $x, y \in M$ with $z \in x.y$ such that $\mu(z) \vee 0.5 < \mu(y) = t$, then $t \in (0.5, 1], \mu(z) < t, y \in \mu_t$. Since $y \in \mu_t$ and μ_t is a left H_v-submodule , $x.y \subseteq \mu_t$ and $\mu(z) \geq t$, for all $z \in x.y$, which is in contradiction with $\mu(z) < t$. Therefore $\mu(y) \geq \mu(z) \vee 0.5$ for all $y \in M$ with $z \in x.y$, which implies $\mu(y) \geq \underset{z\in x.y}{\wedge} (\mu(z) \vee 0.5)$, for all $x, y \in M$. Hence (3) holds.

$(\Leftarrow)$: Assume that $t \in (0.5, 1]$ and $x, y \in \mu_t$. Then $0.5 < t \leq \mu(x) \wedge \mu(y) \leq \underset{z\in x+y}{\wedge} (\mu(z) \vee 0.5)$. It follows that for every $z \in x + y, 0.5 < t \leq \mu(z) \vee 0.5$ and so $t \leq \mu(z)$, which implies $z \in \mu_t$. Hence $x + y \subseteq \mu_t$. Now, we prove the reproducibility rule. Let $x, a \in \mu_t$. Then by condition (2), there exists $y \in M$ such that $x \in a + y$ and $\mu(a) \wedge \mu(x) \leq \mu(y) \vee 0.5$. We show that $y \in \mu_t$. We have $0.5 < t \leq \mu(x) \leq \mu(a) \wedge \mu(x) \leq \mu(y) \vee 0.5$. It follows that $0.5 \leq \mu(y)$ and so $y \in \mu_t$. Therefore $\mu_t = a + \mu_t$, for all $a \in \mu_t$. Similarly, we have $\mu_t + a = \mu_t$, for all $a \in \mu_t$.

Now, assume that $t \in (0.5,1], y \in \mu_t$ and $x \in M$. Then $0.5 < t \le \mu(y) \le \wedge_{z \in x.y} (\mu(z) \vee 0.5)$. It follows that for every $z \in x.y, 0.5 < t \le \mu(z) \vee 0.5$ and so $t \le \mu(z)$, which implies $z \in \mu_t$. Hence $x.y \subseteq \mu_t$. Therefore μ_t is a left H_v-submodule of M for all $t \in (0.5,1]$.

In [24], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup. Based on [24], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy H_v-submodule with thresholds as follows:

Definition 4.6. Let $r, s \in [0,1]$ and $r < s$. Let μ be a fuzzy subset of an H_v-module M. Then μ is called a fuzzy left (right) H_v-submodule with thresholds (r,s) of M if

$(1)\ \mu(x) \wedge \mu(y) \wedge s \le \wedge_{z \in x+y} (\mu(z) \vee r)$, for all $x, y \in M$;

(2) for all $x, a \in M$ there exists $y, z \in M$ such that $x \in (a+y) \cap (z+a)$ and

$\mu(a) \wedge \mu(x) \wedge s \le \mu(y) \vee \mu(z) \vee r$;

$(3)\ \mu(y) \wedge s \le \wedge_{z \in x.y} (\mu(z) \vee r)$, for all $x, y \in M$

$\mu(x) \wedge s \le \wedge_{z \in x.y} (\mu(z) \vee r)$, for all $x, y \in M$.

If μ is a fuzzy left (right) H_v-submodule with thresholds of M, then we can conclude that μ is an ordinary fuzzy left (right) H_v-submodule when $r = 0, s = 1$ and μ is an $(\in, \in \vee q)$-fuzzy left (right) H_v-module when $r = 0, s = 0.5$.

Now, we characterize fuzzy left (right) H_v-submodule with thresholds by their level left (right) H_v-submodule.

Theorem 4.7. A fuzzy subset μ of an H_v-module M is a fuzzy left (right) H_v-submodule with thresholds (r,s) of M if and only if $\mu_t (\ne \phi)$ is a left (right) H_v-submodule of M for all $t \in (r,s]$.

Proof. Let μ be a fuzzy left H_v-submodule with thresholds of M and $t \in (r,s]$. Let $x, y \in \mu_t$. Then $\mu(x) \ge t$ and $\mu(y) \ge t$. Now $\wedge_{z \in x+y} (\mu(z) \vee r) \ge \mu(x) \wedge \mu(y) \wedge s \ge t \wedge s \ge t > r$. So for every $z \in x+y$ we have $\mu(z) \vee r \ge t > r$ which implies $\mu(z) \ge t$ and $z \in \mu_t$. Hence $x + y \subseteq \mu_t$. Now, let $x, a \in \mu_t$, then there exists $y \in M$ such that $x \in a + y$ and $\mu(a) \wedge \mu(x) \wedge s \le \mu(y) \vee r$. From $x, a \in \mu_t$, we have $\mu(x) \ge t$ and $\mu(a) \ge t$, and so $r < t \le t \wedge s \le \mu(a) \wedge \mu(x) \wedge s \le \mu(y) \vee r$, which implies $\mu(y) \ge t$, and so $y \in \mu_t$. Therefore we have $\mu_t = a + \mu_t$ for all $a \in \mu_t$. Similarly we get $\mu_t + a = \mu_t$ for all $a \in \mu_t$. Now, let $y \in \mu_t$ and $x \in M$. Then $\mu(x) \ge t$, and so $\wedge_{z \in x.y} (\mu(z) \vee r) \ge \mu(x) \wedge s \ge t \wedge s \ge t > r$. So for every $z \in x.y$ we have $\mu(z) \vee r \ge t > r$ which implies $\mu(z) \ge t$ and $z \in \mu_t$. Hence $x.y \subseteq \mu_t$. Therefore μ_t is a left H_v-submodule of M, for all $t \in (r,s]$.

Conversely, let μ be a fuzzy subset of M such that $\mu_t (\ne \phi)$ is a left H_v-submodule of M for all $t \in (r,s]$. If there exist $x, y, z \in M$ with $z \in x + y$ such that $\mu(z) \vee r < \mu(x) \wedge \mu(y) \wedge s = t$. Then $t \in (r,s], \mu(z) < t, x \in \mu_t$ and $y \in \mu_t$. Since μ_t is a left H_v-submodule of M and $x, y \in \mu_t$ so $x + y \subseteq \mu_t$. Hence $\mu(z) \ge t$ for all $z \in x + y$. This is in contradiction with $\mu(z) < t$. Therefore $\mu(x) \wedge \mu(y) \wedge s \le \mu(z) \vee r$, for all $x, y, z \in M$ with $z \in x + y$, which implies $\mu(x) \wedge \mu(y) \wedge s \le \wedge_{z \in x+y} (\mu(z) \vee r)$, for all $x, y \in M$. Hence condition (1) of Definition 4.6 holds.

Now, assume that there exist $x_0, a_0 \in M$ such that for all $y, z \in M$ which satisfies $x_0 \in (a_0 + y) \cap (z + a_0)$, the following inequality holds:

$\mu(y) \vee \mu(z) \vee r < \mu(a_0) \wedge \mu(x_0) \wedge s = t$.

Then $t \in (r,s], x_0 \in \mu_t, a_0 \in \mu_t$ and $\mu(y) < t, \mu(z) < t$. Since $x_0, a_0 \in \mu_t$ and μ_t is a left H_v-submodule, so there exists $y_0, z_0 \in \mu_t$ such that $x_0 \in (a_0 + y_0) \cap (z_0 + a_0)$. From $y_0, z_0 \in \mu_t$, we get $\mu(y_0) \ge t, \mu(z_0) \ge t$. This is in contradiction with $\mu(y_0) < t, \mu(z_0) < t$. Therefore $\mu(a) \wedge \mu(x) \wedge s \leq \mu(y) \vee \mu(z) \vee r$. Hence the second condition of Definition 4.6 holds.

If there exist $x, y, z \in M$ with $z \in x.y$ such that $\mu(z) \vee r < \mu(x) \wedge \mu(y) \wedge s = t$, then $t \in (r, s], \mu(z) < t, y \in \mu_t$. Since μ_t is a left H_v-submodule of M and $x \in \mu_t$, so $x.y \subseteq \mu_t$. Hence $\mu(z) \geq t$ for all $z \in x.y$. This is in contradiction with $\mu(z) < t$. Therefore $\mu(y) \wedge s \leq \mu(z) \vee r$, for all $x, z \in M$ with $z \in x.y$, which implies $\mu(y) \wedge s \leq \bigwedge_{z \in x.y} (\mu(z) \vee r)$, for all $x \in M$. Hence condition (3) of Definition 4.6 holds.

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