Characterstics of Ternary Semirings

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Abstract— **In this paper we investigate and study the notion of ternary semi ring as well as Boolean Ternary semiring and characterize them.**

Index Terms—: **b-lattice, Boolean ternary semiring, multiplicatively idempotent, positive rational domain, zeroid**

I. INTRODUCTION

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. The theory of ternary algebraic systems was introduced by D. H. Lehmer [6]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. D. Madhusudhana Rao[8] characterized the primary ideals in ternary semigroups. about T. K. Dutta and S. Kar [4] introduced and studied some properties of ternary semirings which is a generalization of ternary rings. D. Madhusudhana Rao and G. Srinvasa Rao [9, 10] introduced some special element in ternary semiring and studied about ternary semirings. Our main purpose in this paper is to characterize the ternary semirings.

II. PRELIMINARIES

DEFINITION II.1 : A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a *ternary semiring* if T is an additive commutative semigroup satisfying the following conditions :
i) $[[abc]de] = [a[bcd]e] = [ab[cde]]$,
ii) $[(a + b)cd] = [acd] + [bcd]$,
iii) $[a(b + c)d] = [abd] + [acd]$,
iv) $[ab(c + d)] = [abc] + [abd]$ for all $a; b; c; d; e \in$ T.
Throughout T will denote a ternary semiring unless otherwise stated.

NOTE II.2 : For the convenience we write $x_1x_2x_3$ instead of $[x_1x_2x_3]$

NOTE II.3 : Let T be a ternary semiring. If A,B and C are three subsets of T , we shall denote the set ABC = $\{\Sigma abc : a \in A, b \in B, c \in C\}$.

NOTE II.4 : Let T be a ternary semiring. If A,B are two subsets of T , we shall denote the set A + B = $\{a + b : a \in A, b \in B\}$.

NOTE II.5 : Any semiring can be reduced to a ternary semiring.

EXAMPLE II.6 : Let T be an semigroup of all $m \times n$ matrices over the set of all non negative rational numbers. Then T is a ternary semiring with matrix multiplication as the ternary operation.

EXAMPLE II.7 : Let $S = \{. . . ,-2i, -i, 0, i, 2i, . . . \}$ be a ternary semiring withrespect to addition and complex triple multiplication.

EXAMPLE II.8 : The set T consisting of a single element 0 with binary operation defined by 0 + 0 = 0 and ternary operation defined by 0.0.0 = 0 is a ternary semiring. This ternary semiring is called the *null ternary semiring* or the *zero ternary semiring.*

EXAMPLE II.9 : The set T = { 0, 1, 2, 3, 4 } is a ternary semiring with respect to addition modulo 5 and multiplication modulo 5 as ternary operation is defined as follows :

$+_5$	0	1	2	3	4
0	0	1	2	3	5
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

DEFINITION II.10 : A ternary semiring T is said to be ***commutative ternary semiring*** provided $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in$ T.

EXAMPLE II.11 : (Z^0, +, .) is a ternary semiring of infinite order which is commutative.

EXAMPLE II.12 : The set 2I of all evern integers is a commutative ternary semiring with respect to ordinary addition and ternary multiplication [ ] defined by $[abc] = abc$ for all $a, b, c \in$ T.

EXAMPLE II.13 : The set C of all complex numbers is a commutative ternary semiring, the addition and ternary

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multiplication of complex numbers being the two ternary semiring compositions.

NOTE II.14 : The set M of all $n \times n$ matrices with their elements as real numbers (rational numbers, complex numbers, integers) is a non-commutative ternary semiring, with respect to addition and ternary multiplication of matrices as the two ternary semiring compositions.

DEFINITION II.15: An element a in a ternary semiring T is said to be an additive zero provided $a + x = x + a = x$ for all $x \in$ T.

DEFINITION II.16: An element a in a ternary semiring T is said to be an ***absorbing*** w.r.t addition provided $a + x = x + a = a$ for all $x \in$ T.

DEFINITION II.17 : A ternary semiring T is said to be ***multiplicatively left cancellative*** (MLC) if $abx = aby$ implies that x = y for all $a, b, x, y \in$ T.

DEFINITION II.18: A ternary semiring T is said to be ***multiplicatively laterally cancellative*** (MLLC) if $axb = ayb$ implies that x = y for all $a, b, x, y \in$ T.

DEFINITION II.19 : A ternary semiring T is said to be ***multiplicatively right cancellative*** (MRC) if $xab = yab$ implies that x = y for all $a, b, x, y \in$ T.

DEFINITION II.20: A ternary semiring T is said to be ***multiplicatively cancellative*** (MC) if it is multiplicative left cancellative (MLC), multiplicative right cancellative (MRC) and multiplicative laterally cancellative (MLLC).

THEOREM II.21: An multiplicative cancellative ternary semiring T is zero divisor free.

DEFINITION II.22 : An element a of a ternary semiring T is said to be ***additive idempotent*** element provided $a + a = a$.

DEFINITION II.23 : An element a of a ternary semiring T is said to be an ***idempotent*** element provided $a^3 = a$.

DEFINITION II.24 : An element a of a ternary semiring T is said to be ***multiplicatively sub-idempotent*** provided $a + a^3 = a$.

DEFINITION II.25 : A ternary semiring T is said to be a ***sub-idempotent ternary semiring*** provided each of its element is sub-idempotent.

III.CHARACTERSTICS OF A TERNARY SEMIRING

DEFINITION III.1: A ternary semiring T is said to be ***b-lattice*** provided T is an idempotent ternary semiring and (T , +) is commutative.

DEFINITION III.2 : The set Z of ternary semiring T is said to be ***zeioid*** of T provided Z = $\{a \in T : a + b = b$ or $b + a = b$ for some $b \in T\}$.

THEOREM III.3: Let T be a multiplicatively sub-idempotent ternary semi-ring. If T contains the multiplicative identity which is also additive identity, then (i) (T, +) is b-lattice. (ii) (T,+) is zeroed if (T, .) is cancellative and (T, +) is left cancellative.

***Proof* :** (i) Suppose T is a multiplicatively sub-idempotent ternary semiring.

i.e., $a + a^3 = a$ for all $a \in T \rightarrow$ (1). $a + a^3 = a \Rightarrow a(ee + a^2) = a$

$\Rightarrow a.a^2 = a \Rightarrow a^3 = a \;\forall\, a \in T. \rightarrow$ (2). From (1) and (2) we have $a + a = a \rightarrow$ (3).

From (2) and (3) we can say that T is an idempotent ternary semiring.

Since $a + a^3 = a, \Rightarrow a + a^3 + a = a + a \Rightarrow a\,(ee + a^2) + a = a + a$

$\Rightarrow a^3 + a = a + a$. By equation (3) of condition (i) we have $a^3 + a = a \rightarrow$ (4).

Therefore from (1) and (4) we get $a + a^3 = a = a^3 + a \Rightarrow$ (T , +) is commutative. Hence T is a b-lattice.

(ii) **Case 1**: Consider $a + a^3 = a$. Adding on both sides with aba, then $a + a^3 + aba = a + aba$. Since (T. +) is left cancellative, then we have $a^3 + aba = aba \Rightarrow a(a + b)a = aba$, then by cancellative w.r.t ternary multiplication we have $a + b = b$. Therefore (T, +) is zeroed.

Case 2 : Again consider $a + a^3 = a$, then by using equation (2) and (3) of condition (i) we have $a + a = a$

$\Rightarrow a + a + b = a + b$, since (T, +) is left cancellative and hence $a + b = b$. Therefore (T, +) is zeroed.

NOTE III.4: The following is the example for the theorem 2.4.

EXAMPLE III.5 : Let T = { e , a , b } and the binary operation +, ternary multiplication **.** defined as follows:

+	e	a	b
e	e	a	b
a	a	a	a
b	b	a	b

.	e	a	b
e	e	a	b
a	a	a	a
b	b	a	b

DEFINITION III.6 : A ternary semiring T is said to be a ***positive rational domain*** **(PRD)** if and only if (T, **.**) is an abelian group.

THEOREM III.7: Let T be multiplicatively sub-idempotent ternary semiring and PRD. Then $b + a^2b = b \;\forall\, a , b \in T$.

EXAMPLE III.8 : Let T = {1, a }, the following is the example for the above theorem III.7

+	1	a
1	1	1
a	1	a

.	1	a
1	1	a
a	a	a

THEOREM III.9 : Let T be a ternary semi-ring in which ‘ e ‘ is multiplicative identity. If (T, +) is zeroed, then $a^n + aba = aba \;\forall\, n$ and $\forall\, a , b \in T$.

***Proof* :** Let $a \in T$ where T is a zeroed. Then there exists $b \in T$ such that

$a + b = b$ (or) $b + a = b \rightarrow$ (1)

Since $e \in T$ is multiplicative identity, then from (1) $e + b = b$ (or)
$b + e = b \rightarrow (2)$
Take $b + e = b \Rightarrow a (b + e) a = aba \Rightarrow aba + aea = aba \rightarrow$ (3)
$\Rightarrow a (a + b) a + aea = aba$
$\Rightarrow a^3 + aba + aea = aba$, by equation (3) we have $a^3 + aba = aba \rightarrow$ (4). Using the induction it is easy to prove
$a^n + aba = aba$ $\forall n$ and $\forall a, b \in$T.

III. BOOLEAN TERNARY SEMIRING

DEFINITION IV.1: A ternary semiring T is said to be a **Boolean ternary semiring** provided $a^3 = a$ for every $a \in T$

NOTE IV.2: Let T be a ternary semiring T and every element of T is a ternary idempotent, then T is called Boolean Ternary Semiring.

EXAMPLE IV.3 : Let T be a non-empty set and A, B $\in$ P(T). If + and the ternary multiplication. are defined in P(T) as $A + B = (A \quad B') \quad (A' \quad B)$ **and A.B.C = A∩B∩C then Show that (P(T), +, .) is a Boolean ternary semi-ring.**

THEOREM IV.4: If T is a Boolean ternary semiring, then (i) $a + a = 0$ for all $a \in$ T. (ii) $a + b = 0$ implies $a = b$. (iii) $aba = bab$.

***Proof*:** (i) $a \in$ T $\Rightarrow a + a \in$ T. Since $a^3 = a$ for all $a \in$ T, we have $(a + a)^3$
$= (a + a) \Rightarrow (a + a)\ (a + a)\ (a + a) = a + a \Rightarrow a^3 + a^3 + a^3 + a^3 + a^3 + a^3 + a^3 + a^3$
$= a + a \Rightarrow a + a + a + a + a + a + a + a = a + a \Rightarrow (a + a) + (a + a) + (a + a) +$
$(a + a) = a + a \Rightarrow (a + a) + (a + a) +$
$(a + a) + (a + a) = (a + a) + 0 + 0 + 0$, then by using left cancellation law we have $(a + a) = 0$.
(ii) For $a, b \in$ T, $a + b = 0 \Rightarrow a + b = 0 = a + a \Rightarrow a = b$ (by usingcondition(i)).
(iii) Let $a , b \in T$ then $a + b \in T$
$\Rightarrow (a + b)^3 = (a + b)$
$\Rightarrow (a + b) (a + b) (a + b) = (a + b)$
$\Rightarrow (a + b) (a^2 + ab + ba + b^2)$
$= (a + b) \Rightarrow a^3 + aab + aba + ab^2$
$+ ba^2 + bab + bba + bb^2 = (a + b)$
$\Rightarrow a^3 + aab + aba + abb + baa + bab$
$+ bba + b^3 = (a + b)$
$\Rightarrow a + (aab + aab) + (bab + bab) + (aba + bab) + b = a + b \rightarrow$ (1)
$\Rightarrow a + ab(a + a) + ba(b + b) + (aba + bab) + b = a + b$
Since $a + a = 0$ and cancellation laws, we have $aba + bab = 0 \rightarrow$ (2)
By condition (ii) i.e. $a + b = 0$ then
$a = b$ then eq.(2) becomes $aba = bab$.

THEOREM IV.5 : A Boolean ternary semi-ring T and $a, b \in$ T such that $aba = 0$ then $bab = 0$ and $asb = 0$ $\forall s \in T$.

***Proof*:** Consider T is a Boolean ternary semi-ring. We know that every Boolean ternary semi-ring has zero elements. i.e., $a^n = 0$ for odd natural number n implies
a =0. Suppose that $a, b \in$ T such that $aba = 0$. Consider $(bab)^3 = bab.bab.\ bab = bab.abbbab = 0$.
Then $(bab)^3 = 0$ implies $bab = 0$.
Let s $\in$ T. Consider $(asb)^3$
$= asb.asb.asb = sab.asbasb = 0$.
Then $(asb)^3 = 0$ implies $asb = 0$ for all $s \in$ T.

THEOREM IV.6 : Let (T, +, .) be a zeroid ternary semi-ring with multiplicative identity ' e ' and cancellative law holds then T is a Boolean ternary semiring and
$(a + b)^n = (a + b)$ for n is odd positive integer.

***Proof*:** Assume that T is a zeroed ternary semi-ring. Then for $a \in$ T, there exists
$b \in$ T such that $a + b = b$ or $b + a = b$. Since T zeroid and contains multiplicative identity e implies $a + e = e$ and $a + a = a$. Suppose that $a = aee$
$\Rightarrow a + a = a (a + e) e \Rightarrow a + a = a^2e + aee \Rightarrow a + a = a^2(a + e) + a \Rightarrow a + a = a^3 + a$. Therefore by using cancellative property $a = a^3$ and hence T is Boolean Ternary Semiring.
Consider $(a + b)^3 = (a + b)(a + b)$
$(a + b) = (a + b)(a^2 + ab + ba + b^2) = a^3 + aab + aba + abb + baa + bab + bba + b^3 = a + b + (aab + bab) + (aba + bba) + abb + baa = a + b + (a + b)ab + (a + b)ba + abb + baa = a + b + bab + bba + abb + baa = a + b + ba(a + b) + bba + abb = a + b + bab + bba + abb$
$= a(ee + bb) + b(ee + ab) + bba = aee + bee + bba = a + b(ee + ba) = a + b$. Therefore $(a + b)^3 = (a + b)$.
Again $(a + b)^5 = (a + b)^3(a + b)(a + b) = (a + b)(a + b)(a + b) = (a + b)^3 = a + b$.
Continuing like this we have $(a + b)^n =$
$a + b$.

DEFINITION IV.7: Suppose $p>2$ is prime number. A ternary semi-ring T is called ***Prime ternary semiring*** or simply
***p*-ternary semi-ring** provided that for all $a \in T$, $a^p = 0$ and $pa = 0$.

CONCLUSION

In this paper mainly we studied about characteristics of ternary semirings.

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