ON SOME HSU-STRUCTURE AND H-STRUCTURE MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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Abstract—In this paper I have studied Hsu-structure & H-structure manifolds and obtained with constant holomorphic sectional curvature

Index Terms— H-structure manifold , Hsu-structure .Curvature tensor almost complex , almost product and almost tangent structures.
Mathematical Classification : 53C15, 53C25

I. INTRODUCTION
Structures on differentiable manifolds by introducing vector-valued linear functions satisfying some algebraic equations have been studied by a number of mathematicians. K. L. Duggal in [1] defined on a differentiable manifold Hsu-structure which is more general than almost complex , almost product and almost tangent structures.
Let there be defined on $V_n$, a vector valued linear function $F$ of class $C^\infty$ such that

$$F^2 = a^r I_n, \quad 0 \leq r \leq n$$

where r is an integer and a is real or imaginary number. Then $F$ is called Hsu-structure and $V_n$ is called the Hsu-structure manifold.

Let $M^n$ be a $n$-dimensional differentiable manifold of class $C^\infty$. A vector-valued linear function $F$ of class $C^\infty$ is defined on $M$ such that

$$F^2 (X) = a^r X \quad (1.1)$$

where $X$ is an arbitrary vector field and $a$ is any real or purely imaginary number. Then $F$ is said to give a differentiable structure called Hsu-structure on $M$ Defined by (1.1). If $a^{\frac{r}{2}} \neq 0$ we have the known $\pi$-structure [3], if $a^{\frac{r}{2}} = 0$ we have an almost tangent structure. For $a^{\frac{r}{2}} = \pm 1$ or $a^{\frac{r}{2}} = \pm \sqrt{-1}$ we obtain an almost product or almost complex structure respectively.

Suppose further that $M$ admits a Hermitian metric $g$ satisfying

$$g(\overline{X}, \overline{Y}) + a^r g(X, Y) = 0 \quad (1.2)$$

where $\overline{X} = FX$ and $X, Y$ are vector fields on $M$. Then , we say that $(g, F)$ gives to $M$ an H-structure and $M$ is called K-manifold.

If the structure tensor $F$ is parallel (i.e. $(\nabla_X F)Y = 0$ where $\nabla$ is the Riemannian connection ), then $M$ is called K-manifold.

An H-structure manifold $M$ will be called nearly K-manifold (briefly NK-manifold) if the structure tensor $F$ satisfies the condition $(\nabla_X F)X = 0$, for arbitrary vector field $X$ on $M$.
In the present article we deal with some 2m-dimensional H-structure manifolds. In the next paragraph we shall study an H-structure manifold admitting pointwise constant holomorphic sectional curvature . In the later we obtain the main result of the present paper on NK-manifolds.

II. ON H-STRUCTURE MANIFOLDS
On a 2m-dimensional H-structure manifold $M$ we consider a $(0, 2)$ tensor such that

$$\phi(X, Y) = g(\overline{X}, Y) = -g(X, \overline{Y}) \quad (2.1)$$

It is easy to prove that the following results

$$\phi(X, Y) + \phi(Y, X) = 0 \quad (2.2)$$

$$\phi(\overline{X}, Y) + a^r \phi(X, Y) = 0 \quad (2.3)$$

$$(\nabla_X \phi)(Y, Z) + (\nabla_Y \phi)(Z, Y) = 0 \quad (2.4)$$

$$(\nabla_X \phi)(\overline{Y}, \overline{Z}) = a^r (\nabla_X \phi)(Y, Z) \quad (2.5)$$

We denote by $(W, X, Y, Z) = g((\nabla_Y F)X, \nabla_Z F)Z$ and because of (2.2), (2.3) we obtain

$$(W, X, Y, Z) = (Y, Z, W, X) \quad (2.6)$$

We assume that the curvature tensor $R$ is defined by

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\[ R(X, Y)Z = \nabla_{[X, Y]} Z - [\nabla_X, \nabla_Y] Z \]

and

\[ R(W, X, Y, Z) = g(R(W, X)Y, Z) \]

for arbitrary vector fields W, X, Y and Z on M.

The holomorphic sectional curvature H(x) is defined by

\[ H(x) = R(x, \bar{x}, x, \bar{x}) / g(x, x)g(\bar{x}, \bar{x}) \quad (2.7) \]

for \( x \in T_p(M), (p \in M) \) where \( T_p(M) \) is the tangent space of M at p.

**Theorem 2.1**

Let M be an H-structure manifold of pointwise constant holomorphic sectional curvature \( c(p) \). Then

\[
4a' c(p)[2\phi(x, y)\phi(z, w) - \phi(x, w)\phi(y, z) + \phi(x, z)\phi(y, w) + a' g(x, w)g(y, z) - a' g(x, z)g(y, w)]
\]

\[
= -3a' R(\bar{w}, x, y, z) - 3R(\bar{w}, x, y, z) + a' R(\bar{w}, \bar{x}, y, z) + a' R(\bar{w}, x, y, \bar{z}) - a' R(\bar{w}, x, y, \bar{z})
\]

\[
+ 3a' R(\bar{w}, x, y, z) + 3a' R(\bar{w}, \bar{x}, y, z) - a' R(\bar{w}, x, y, \bar{z}) + 3a' R(\bar{w}, y, \bar{x}, z) + 3R(\bar{w}, y, \bar{x}, z)
\]

\[
- a' R(\bar{w}, y, z) + a' R(\bar{w}, y, \bar{x}, \bar{z}) - a' R(\bar{w}, y, z) - 3a' R(\bar{w}, y, \bar{x}) - 3a' R(\bar{w}, y, z)
\]

\[
+ a' R(\bar{w}, y, \bar{x}, z)
\]

\[ (2.8) \]

**Proof:** Since \( H(x) = c(p) \), (2.7) takes the form

\[ R(x, \bar{x}, x, \bar{x}) = c(p)g(x, x)g(\bar{x}, \bar{x}) \quad (2.9) \]

By linearizing (2.9) and using Binachi identity, we get

\[
4a' c(g(x, y)g(z, w) + g(x, z)g(y, w) + g(x, w)g(y, z)
\]

\[
= R(\bar{w}, x, y, z) - 2R(\bar{w}, x, y, z) + R(\bar{w}, \bar{x}, y, z) + R(\bar{w}, x, y, \bar{z}) - 2R(\bar{w}, x, y, \bar{z})
\]

\[
- 2R(\bar{w}, x, y, z) + R(\bar{w}, \bar{x}, y, z) + R(\bar{w}, y, \bar{x}, z) - 2R(\bar{w}, y, \bar{x}, z)
\]

\[
+ R(\bar{w}, y, z) + R(\bar{w}, y, \bar{x}, \bar{z}) - 2R(\bar{w}, y, z) + R(\bar{w}, y, x, z)
\]

\[ (2.10) \]

In (2.10) we replace Y and W by \( \bar{Y} \) and \( \bar{W} \) and in the resulting equation we replace X and Y by Y and X respectively. Adding the last two equations we obtain (2.8).

We can choose an orthonormal frame field \( \{E_1, \ldots, E_m, E_{m+1}, \ldots, E_{2m}\} \) such that \( E_{m+i} = \sqrt{-1}E_i / a^{i/2}, i = 1, \ldots, m \).

We denote by \( k^* \) and \( k^* \) the Ricci tensor and the \( \text{Ricci}^* \) tensor of \( M \), respectively. The \( \text{Ricci}^* \) tensor \( k^* \) is defined by

\[ k^*(x, y) = \text{traceof}(z \mapsto R(\bar{z}, x)y) \]

for \( x, y, z \in T_p(M) \).

**Lemma 2.2**

If M is an H-structure manifold and \( \{E_i\} \) is an orthonormal frame field, for arbitrary vector fields X, Y on M we have
\[
\sum_{i=1}^{2m} R(X, E_i, Y, E_i) = -a' \sum_{i=1}^{2m} R(X, E_i, Y, E_i)
\]
\[
\sum_{i=1}^{2m} R(X, E_i, Y, \overline{E}_i) = -\sum R(X, \overline{E}_i, Y, E_i)
\]

**Proof:** The proof depends on the above way of the determination of the orthogonal frame field \( \{E_i\} \).

We can easily prove the following.

**Lemma 2.3**

Let \( M \) be an H-structure manifold. Then for arbitrary vector fields \( X, Y \) on \( M \). We have

\[
k(X, Y) = k(Y, X), \quad k^*(X, \overline{Y}) = -a'k^*(Y, X), \quad k^*(\overline{X}, Y) = -k^*(\overline{Y}, X)
\]

We denote by \( \delta \) and \( s^* \) the scalar and the "scalar" curvature of \( M \) respectively. Then, using the theorem 2.1 and the lemma 2.2 and 2.3 we obtain.

**Proposition 2.4**

Let \( M \) be a \( 2m \)-dimensional H-structure manifold of pointwise constant holomorphic sectional curvature \( c(p) \). Then, for arbitrary vector fields \( X, Y \) on \( M \), we have

\[
a'k(X, Y) - k(\overline{X}, \overline{Y}) - 3[k^*(X, Y) + k^*(Y, X)] = 4(m + 1)c(p)a' g(X, Y)
\]

\[
a' \delta - 3s^* = 4m(m + 1)a' c(p)
\]

The main result of the second paragraph ( theorem 2.1 and proposition 2.4) for \( a' = -1 \) have been obtained by G.B.Rizza in [4] (fundamental identity (11) and theorem 1).

3. ON NEARLY K-MANIFOLDS

We denote by \((W, X, Y, Z) = g((\overline{\nabla}_Y F)X, (\overline{\nabla}_X F)Z)\). By definition of the NK-manifold and the curvature tensor \( R \) we obtain that : \( R(W, X, Y, Z) - R(W, X, \overline{Y}, \overline{Z}) \) depends on the quantities : \((W, X, Y, Z), (W, Y, X, Z), (W, Z, X, Y), (W, X, Y, \overline{Z}), (W, Y, X, \overline{Z})\) and \((W, Z, X, \overline{Y})\). Applying the fundamental properties of \( R(W, X, Y, Z) \) we obtain .

**Proposition 3.1**

Let \( M \) be a NK-manifold. If \( W, X, Y \) and \( Z \) are arbitrary vector fields on \( M \), then

\[
R(W, X, Y, Z) = \frac{1}{a' - 3} [2(W, X, Y, Z) - (W, Y, X, Z) - (W, Z, X, Y)]
\]

\[
R(W, X, \overline{Y}, \overline{Z}) = \frac{1}{a' - 3} a'^2 [2(W, X, Y, Z) - (W, Y, X, Z) + (W, Z, X, Y)]
\]

Using the proposition 3.1 and the definitions of the Ricci tensor and the Ricci tensor we get the following.

**Lemma 3.2**

For arbitrary vector fields \( X \) and \( Y \) on a NK-manifold it holds :

\[
k(X, Y) = \frac{1}{a' - 3} \sum_{i=1}^{2m} (X, E_i, Y, E_i)
\]

\[
k(\overline{X}, \overline{Y}) = -a'k(X, Y), \quad k^*(X, Y) = k^*(Y, X)
\]

\[
k^*(X, Y) = \frac{1}{a' - 3} a'k(X, Y)
\]
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By virtue of the first relation of proposition 2.4, the lemma 3.2 and [2] (p.292) we can obtain the main result:

**Theorem 3.3**

If $M$ is a 2m-dimensional connected NK-manifold of pointwise constant holomorphic sectional curvature, then $M$ is an Einstein manifold.

For NK-manifolds of small dimension we can state the following.

**Proposition 3.4**

A NK-manifold $M$ of dimension $n = 2,4$ is a K-manifold.

**Proof:**

It is clear that a 2-dimensional NK-manifold is a K-manifold.

If $M$ is a 4-dimensional NK-manifold, we choose an orthonormal frame field on an open subset of $M$ to be of the form

$$\begin{pmatrix}
    E_1, E_2 \\
    \frac{\sqrt{-1}}{a^{1/2}} E_1, \frac{\sqrt{-1}}{a^{1/2}} E_2
\end{pmatrix}.$$

We can easily prove that $(\nabla_{E_i} F)E_2$ is perpendicular to $E_1$ and $E_2$. Because of

$$(\nabla_X \phi)(Y, Z) = \alpha (\nabla_X \phi)(Y, Z) = -\alpha (\nabla_X \phi)(Z, Y)$$

It is proved that $(\nabla_{E_i} F)E_2$ is perpendicular to $\frac{\sqrt{-1}}{a^{1/2}} E_1$ and $\frac{\sqrt{-1}}{a^{1/2}} E_2$.

Hence

$$(\nabla_{E_i} F)E_j = 0, \quad (i, j = 1,2)$$

**REFERENCES**


