ON SOME HSU-STRUCTURE AND H-STRUCTURE MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

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***Abstract*—In this paper I have studied Hsu-structure & H-structure manifolds and obtained with constant holomorphic sectional curvature**

***Index Terms*— H-structure manifold , Hsu-structure ,Curvature tensor almost complex , almost product and almost tangent structures.**
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I. INTRODUCTION

Structures on differentiable manifolds by introducing vector-valued linear functions satisfying some algebraic equations have been studied by a number of mathematicians. K. L. Duggal in [1] defined on a differentiable manifold Hsu-structure which is more general than almost complex , almost product and almost tangent structures.

Let there be defined on V_n , a vector valued linear function F of class C such that

$$F^2 = a^r I_n \qquad 0 \le r \le n$$

where r is an integer and a is real or imaginry number. Then F is called Hsu – structure and V_n is called the **Hsu – structure manifold**.

Let M be a n-dimensional differentiable manifold of class C^∞ . A vector-valued linear function F of class C^∞ is defined on M such that

$$F^2(X) = a^r X \qquad (1.1)$$

where X is an arbitrary vector field and a is any real or purely imaginary number. Then F is said to give a differentiable structure called Hsu-structure on M Defined by (1.1). If $a^{r/2} \neq 0$ we have the known π -structure [3], if $a^{r/2} = 0$ we have an almost tangent structure. For $a^{r/2} = \pm 1$ or $a^{r/2} = \pm\sqrt{-1}$ we obtain an almost product or almost complex structure respectively.

Suppose further that M admits a Hermitian metric g satisfying

$$g(\overline{X}, \overline{Y}) + a^r g(X, Y) = 0 \qquad (1.2)$$

where $\overline{X} = FX$ and X, Y are vector fields on M. Then , we say that (g, F) gives to M an H-structure and M is called K-manifold.

If the structure tensor F is parallel (i.e. $(\nabla_X F)Y = 0$ where ∇ is the Riemannian connection), then M is called K-manifold.

An H-structure manifold M will be called nearly K-manifold (briefly NK-manifold) if the structure tensor F satisfies the condition $(\nabla_X F)X = 0$, for arbitrary vector field X on M.

In the present article we deal with some 2m-dimensional H-structure manifolds. In the next paragraph we shall study an H-structure manifold admitting pointwise constant holomorphic sectional curvature . In the later we obtain the main result of the present paper on NK-manifolds.

II. ON H-STRUCTURE MANIFOLDS

On a 2m-dimensional H- structure manifold M we consider a (0, 2) tensor such that

$$\phi(X, Y) = g(\overline{X}, Y) = -g(X, \overline{Y}) \qquad (2.1)$$

It is easy to prove that the following results

$$\phi(X, Y) + \phi(Y, X) = 0 \qquad (2.2)$$

$$\phi(\overline{X}, \overline{Y}) + a^r \phi(X, Y) = 0 \qquad (2.3)$$

$$(\nabla_X \phi)(Y, Z) + (\nabla_X \phi)(Z, Y) = 0 \qquad (2.4)$$

$$(\nabla_X \phi(\overline{Y}, \overline{Z}) = a^r (\nabla_X \phi(Y, Z) \qquad (2.5)$$

We denote by $(W, X, Y, Z) = g((\nabla_W F)X, \nabla_Y F)Z$ and because of (2.2), (2.3) we obtain

$$\begin{aligned} (W, X, Y, Z) &= (Y, Z, W, X) \\ (W, \overline{X}, Y, \overline{Z}) &= -a^r (W, X, Y, Z) \\ (W, \overline{X}, Y, Z) &= -(W, X, Y, \overline{Z}) \end{aligned} \qquad (2.6)$$

We assume that the curvature tensor R is defined by

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$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

and

$$R(W,X,Y,Z) = g(R(W,X)Y,Z)$$

for arbitrary vector fields W,X,Y and Z on M.
The holomorphic sectional curvature H(x) is defined by

$$H(x) = R(x,\bar{x},x,\bar{x}) / g(x,x)g(\bar{x},\bar{x}) \qquad (2.7)$$

for $x \in T_p(M), (p \in M)$ where $T_p(M)$ is the tangent space of M at p.

Theorem 2.1
Let M be an H-structure manifold of pointwise constant holomorphic sectional curvature c(p) . Then

$$\begin{aligned}
&4a^r c(p)[2\phi(x,y)\phi(z,w) - \phi(x,w)\phi(y,z) + \phi(x,z)\phi(y,w) + a^r g(x,w)g(y,z) - a^r g(x,z)g(y,w)] \\
&= -3a^{2r}R(w,x,y,z) - 3R(\bar{w},\bar{x},\bar{y},\bar{z}) + a^r R(\bar{w},\bar{x},y,z) + a^r R(w,x,\bar{y},\bar{z}) - a^r R(\bar{w},x,\bar{y},z) \\
&\quad + 3a^r R(\bar{w},x,y,\bar{z}) + 3a^r R(w,\bar{x},\bar{y},z) - a^r R(w,\bar{x},y,\bar{z}) + 3a^r R(w,y,x,z) + 3R(\bar{w},\bar{y},\bar{x},\bar{z}) \\
&\quad - a^r R(\bar{w},\bar{y},x,z) - a^r R(w,y,\bar{x},\bar{z}) + a^r R(\bar{w},y,\bar{x},z) - 3a^r R(\bar{w},y,x,\bar{z}) - 3a^r R(w,\bar{y},\bar{x},z) \\
&\quad + a^r R(w,\bar{y},x,\bar{z})
\end{aligned} \qquad (2.8)$$

Proof : Since $H(x) = c(p)$, (2.7) takes the form

$$R(x,\bar{x},x,\bar{x}) = c(p)g(x,x)g(\bar{x},\bar{x}) \qquad (2.9)$$

By linearizing (2.9) and using Binachi identity , we get

$$\begin{aligned}
&4a^r c[g(x,y)g(z,w) + g(x,z)g(y,w) + g(x,w)g(y,z) \\
&\quad = R(\bar{w},\bar{x},y,z) - 2R(\bar{w},x,\bar{y},z) + R(\bar{w},x,y,\bar{z}) + R(w,\bar{x},\bar{y},z) \\
&\quad\; - 2R(w,\bar{x},y,\bar{z}) + R(w,x,\bar{y},\bar{z}) + R(\bar{w},\bar{y},x,z) - 2R(\bar{w},y,\bar{x},z) \\
&\quad\; + R(\bar{w},y,x,\bar{z}) + R(w,\bar{y},\bar{x},z) - 2R(w,\bar{y},x,\bar{z}) + R(w,y,\bar{x},\bar{z})
\end{aligned} \qquad (2.10)$$

In (2.10) we replace Y and W by $\bar{Y}$ and $\bar{W}$ and in the resulting equation we replace X and Y by Y and X respectively . Adding the last two equations we obtain (2.8).

We can choose an orthonormal frame field $\{E_1, \ldots\ldots\ldots, E_m, E_{m+1}, \ldots\ldots\ldots\ldots, E_{2m} \text{ such that } E_{m+i} = \sqrt{-1}E_i / a^{r/2}, i = 1, \ldots\ldots\ldots\ldots, m\}$

We denote by k and k^* the Ricci tensor and the $Ricci^*$ tensor of M , respectively. The $Ricci^*$ tensor k^* is defined by

$$k^*(x,y) = traceof(z \to R(\bar{z},x)\bar{y})$$

for $x,y,z \in T_p(M)$.

Lemma 2.2
If M is an H-structure manifold and $\{E_i\}$ is an orthonormal frame field , for arbitrary vector fields X, Y on M we have

$$\sum_{i=1}^{2m} R(X,\overline{E}_i,Y,\overline{E}_i) = -a^r \sum_{i=1}^{2m} R(X,E_i,Y,E_i)$$
$$\sum_{i=1}^{2m} R(X,E_i,Y,\overline{E}_i) = -\sum R(X,\overline{E}_i,Y,E_i)$$

Proof : The proof depends on the above way of the determination of the orthogonal frame field $\{E_i\}$.

We can easily prove the following.

Lemma 2.3

Let M be an H-structure manifold . Then for arbitrary vector fields X, Y on M. We have

$$k(X,Y) = k(Y,X),\ k^*(\overline{X},\overline{Y}) = -a^r k^*(Y,X),\ k^*(\overline{X},Y) = -k^*(\overline{Y},X)$$

We denote by s and s^* the scalar and the $^*scalar$ curvature of M respectively. Then, using the theorem 2.1 and the lemma 2.2 and 2.3 we obtain .

Proposition 2.4

Let M be a 2m –dimensional H-structure manifold of pointwise constant holomorphic sectional curvature c(p). Then , for arbitrary vector fields X, Y on M, we have

$$a^r k(X,Y)\_k(\overline{X},\overline{Y}) - 3[k^*(X,Y) + k^*(Y,X)] = 4(m+1)c(p)a^r g(X,Y)$$
$$a^r s - 3s^* = 4m(m+1)a^r c(p)$$

The main result of the second paragraph (theorem 2.1 and proposition 2.4) for $a^r = -1$ have been obtained by G.B.Rizza in [4] (fundamental identity (11) and theorem 1).

3. ON NEARLY K-MANIFOLDS

We denote by $(W,X,Y,Z) = g((\nabla_W F)X,(\nabla_Y F)Z)$. By definition of the NK-manifold and the curvature tensor R we obtain that : $R(W,X,Y,Z) - R(W,X,\overline{Y},\overline{Z})$ depends on the quantities : $(W,X,Y,Z),(W,Y,X,Z),(W,Z,X,Y),(W,X,Y,\overline{Z}),(W,Y,X,\overline{Z})$ and $(W,Z,X,\overline{Y})$. Applying the fundamental properties of R(W, X, Y, Z) we obtain .

Proposition 3.1

Let M be a NK-manifold. If W, X, Y and Z are arbitrary vector fields on M, then

$$R(W,X,Y,Z) = \frac{1}{a^r - 3}[2(W,X,Y,Z) + (W,Y,X,Z) - (W,Z,X,Y)]$$
$$R(W,X,\overline{Y},\overline{Z}) = \frac{1}{a^r - 3} a^{r/2}[2(W,X,Y,Z) - (W,Y,X,Z) + (W,Z,X,Y)]$$

Using the proposition 3.1 and the definitions of the Ricci tensor and the $Ricci^*tensor$ we get the following .

Lemma 3.2

For arbitrary vector fields X and Y on a NK-manifold it holds :

$$k(X,Y) = \frac{1}{a^r - 3}\sum_{i=1}^{2m}(X,E_i,Y,E_i)$$
$$k(\overline{X},\overline{Y}) = -a^r k(X,Y), \quad k^*(X,Y) = k^*(Y,X)$$
$$k^*(X,Y) = \frac{1}{a^r - 3} a^r k(X,Y)$$

By virtue of the first relation of proposition 2.4, the lemma 3.2 and [2] (p.292) we can obtain the main result :

Theorem 3.3

If M is a 2m-dimensional connected NK-manifold of pointwise constant holomorphic sectional curvature , then M is an Einstein manifold .

For NK-manifolds of small dimension we can state the following .

Proposition 3.4

A NK-manifold M of dimension n = 2,4 is a K-manifold.

Proof:

It is clear that a 2-dimensional NK-manifold is a K-manifold.

If M is a 4-dimensional NK-manifold, we choose an orthonormal frame field on an open subset of M to be of the from $\left\{ E_1, E_2 \frac{\sqrt{-1}}{a^{r/2}}\overline{E}_1, \frac{\sqrt{-1}}{a^{r/2}}\overline{E}_2 \right\}$.

We can easily prove that $(\nabla_{E_1} F)E_2$ is perpendicular to E_1 and E_2. Because of

$$(\nabla_X \phi)(\overline{Y}, \overline{Z}) = a^r (\nabla_X \phi)(Y, Z) = -a^r (\nabla_X \phi)(Z, Y)$$

It is proved that $(\nabla_{E_1} F)E_2$ is perpendicular to $\frac{\sqrt{-1}}{a^{r/2}}\overline{E}_1$ and $\frac{\sqrt{-1}}{a^{r/2}}\overline{E}_2$.

Hence

$$(\nabla_{E_I} F)E_j = 0, \qquad (i, j = 1,2)$$

REFERENCES

[1] K. L. DUGGAL, On differentiable structures defined by algebraic equations, I. Nijenhuis tensor. Tensor N.S. 22 (1971), 238 -242.

[2] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry , I, J. Wiley, New York, 1969.

[3] G. LEGRAND, Sur les varietes a structure de Presque prouit complexe, C.R. Acad. Sc. Paris 242 (1956), 335-337.

[4] G. B. RIZZA, On almost Hermitian manifolds with constant holomorphic curvature at a point, Tensor N.S. 50 (1991), 79-89.