Pythagorean Triangle with Area/Perimeter as a Quartic Integer

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***Abstract*— Patterns of Pythagorean triangles in each of which the ratio Area / Perimeter may be expressed as a quartic integer. A few interesting relations among the sides are also given.**

***Index Terms*—Area/perimeter, Pythagorean triangle, quartic.**

I. INTRODUCTION

The Pythagorean numbers play a significant role in the theory of higher arithmetic as they come in the majority of indeterminate problems; had a marvelous effect on a credulous people and always occupy a remarkable position due to unquestioned historical importance. The method of obtaining three non-zero integers x, y and z under certain relations satisfying the relation $x^2 + y^2 = z^2$ has been a matter of interest to various Mathematicians [1]-[6]. In [7]-[18], special Pythagorean problems are studied. In this communication, we search for patterns of Pythagorean triangles wherein each of which the ratio Area/Perimeter is represented by a quartic integer. In addition to the solutions present in [19], we exhibit some more solutions and few relations among the sides.

Notations:

- $t_{m,n}$ - Polygonal number of rank n with side m.
- $cp_{m,n}$ – Centered polygonal number of rank n with side m.
- $CP_{m,n}$ – Centered pyramidal number of rank n with side m
- $G(n)$ - Gnomonic number of rank n.

II. METHOD OF ANALYSIS

The most cited solution of the Pythagorean equation,

$$x^2 + y^2 = z^2 \qquad (1)$$

is represented by

$$x = 2mn\,;y = m^2 - n^2\,;z = m^2 + n^2 \qquad (2)$$

Denoting the Area and Perimeter of the Pythagorean triangle by A and P respectively, the assumption

$$\frac{A}{P} = \alpha^4\ ,\ \alpha > 1$$

leads to the equation

$$n(m - n) = 2\alpha^4 \qquad (3)$$

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It is noted that (1) can be satisfied by the following triple of integers:

$$(12\alpha^4, 1 + 4\alpha^4, 1 + 4\alpha^4 + 8\alpha^8)\,,$$
$$(\ 4\alpha^4 + 8, 4\alpha^4 + \alpha^8, 4\alpha^4 + \alpha^8 + 8)\,,$$
$$(2\alpha^4(\alpha^4 + 2), 4\alpha^4 + 4, 2\alpha^8 + 4\alpha^4 + 4),$$
$$(4\alpha^4(1 + 2\alpha^2), \alpha^2(1 + 4\alpha^2), \alpha^2(1 + 4\alpha^2) + 8\alpha^6),$$
$$(2\alpha^2(2\alpha^2 + 1), 4\alpha^4(\alpha^2 + 1), 2\alpha^2(2\alpha^4 + 2\alpha^2 + 1)),$$
$$(2(2\alpha^4 + 1), 4\alpha^4(2\alpha^4 + 1), 4\alpha^4(2\alpha^4 + 1) + 2).$$

In addition to the above solutions, we obtain the different patterns of solutions to (1).

A. Pattern 1

Equation (3) can be written in the ratio form as

$$\frac{n}{\alpha^2} = \frac{2}{m-n} = \frac{c}{d}$$

Using the method of cross multiplication, we obtain

$$m = 2\frac{d}{c} + \frac{c}{d}\alpha^4$$

$$n = \frac{c}{d}\alpha^4$$

Since m and n are to be integers, we choose

$$\alpha = d\beta \ \text{ and } \ d = c\delta \quad ,(\beta, \delta \neq 0).$$

Therefore

$$m = 2\delta + cd^3\beta^4 \quad ; \quad n = cd^3\beta^4 \qquad (4)$$

Using (4) in (2) , the sides of the Pythagorean triangle are obtained as

$$x = 2(2\delta + cd^3\beta^4)(cd^3\beta^4)$$
$$y = 4\delta(\delta + cd^3\beta^4)$$
$$z = 4\delta^2 + 2c^2\beta^8 d^6 + 4cd^3\beta^4\delta$$

Properties

- $x(1, d, 1,1) - G(2d^3)$ is a centered square number of rank d^3.
- $y(1, d, 1,1) - 4CP^6_{d^3}$ is a perfect square.
- $z(1, d, 1,1) - y(1, d, 1,1)$ is a Kynea prime.
- $d^3[x(1, d, 1,1) - y(1, d, 1,1)] + 3CP^6_d$ is a Stella Octangula number of rank d^3.
- $z(1, d, 1,1) - x(1, d, 1,1) + 2t_{4,d^3}$ is a nasty number.

B. Pattern II

We write (3) in another ratio as

$$\frac{n}{\alpha} = \frac{2\alpha^3}{m - n} = \frac{c}{d}$$

Using the method of cross multiplication, we obtain

$$m = 2\frac{d}{c}\alpha^3 + \frac{c}{d}\alpha$$
$$n = \frac{c}{d}\alpha$$

Assuming $\alpha = cd$, we get m and n as integers as follows:

$$m = c^2(2d^4 + 1) \quad ; \quad n = c^2 \qquad (5)$$

Using (5) in (2), the sides of Pythagorean triangle are obtained as

$$x = 2c^4(1 + 2d^4)$$

$$y = 4c^4d^4(1 + d^4)$$
$$z = 2c^4(1 + 2d^4 + 2d^8)$$

Properties

- The following expressions are nasty numbers:
  (a) $x(c,1) + y(c,1) + z(c,1)$
  (b) $6[z(c,1) - x(c,1)]$
  (c) $6[z(1,d) - y(1,d) + x(1,d) - t_{4,2d^2}]$
- $z(1,d) - y(1,d)$ is a Kynea prime.
- $z(1,d) - 1$ is a centered octagonal number of rank d^4.
- Note that the pair (x,z) satisfies the parabola $x^2 = 4c^4(z - c^4)$.
- Note that the pair (x,y) satisfies the parabola $x^2 = 4c^4(y + c^4)$.

C. Pattern III

Equation (3) may also be expressed in the form of ratio as

$$\frac{n}{\alpha^2} = \frac{2\alpha^2}{m - n} = \frac{c}{d}$$

Following procedure in above two patterns we *m* and *n* as

$$m = \frac{\alpha^2}{cd}(2d^2 + c^2)$$

$$n = \frac{c}{d}\alpha^2$$

As we interested only in integer solutions we assume $\alpha^2 = cd$ to obtain *m* and *n* as integers.
Therefore,

$$m = 2d^2 + c^2 \quad ; \quad n = c^2 \qquad (6)$$

Using (6) in (2), the sides of Pythagorean triangle are obtained as

$$x = 2c^2(2d^2 + c^2)$$
$$y = 4d^2(d^2 + c^2)$$
$$z = 4d^2(d^2 + c^2) + 2c^4$$

Properties

- Each of the following expressions are perfect squares
  (a) $z - x$.
  (b) $y + z - x - 8t_{4,d^2}$.
- $3(z - y)$ is a nasty number.
- Each of the following expressions are kynea primes:
  (a) $z(1,d) - y(1,d)$.
  (b) $z(1,d) - 8t_{3,d^2}$.

CONCLUSION

To conclude, one may search for other choices of Pythagorean triangles in connection with special figurate numbers and their corresponding properties.

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