ON THE CLASSIFICATION OF THE ALMOST r-CONTACT METRIC MANIFOLDS

Virendra Nath Pathak, Geeta Verma

***Abstract*— ABSTRACT : The vector space of the tensors $\Im$ of type (0,3) having the same symmetries as the covariant derivative of the fundamental form of an almost r-contact metric manifold is considered. A scheme of decomposition of $\Im$ into orthogonal components which are invariant under the action of $U(n)\times 1$ is given . Using this decomposition there are found 12 natural basic classes of almost r-contact metric manifolds. The classes of cosymplectic, $\alpha-$Sasakian, $\alpha-$ Kenmotsu, etc. manifolds fit nicely to these considerations. On the other hand , many new interesting classes of almost r-contact metric manifolds arise**

***Index Terms*— almost r-contact metric manifold, cosymplectic, $\alpha-$Sasakian , $\alpha-$Kenmotsu, Covariant derivative of the fundamental form and decomposition of a space of tensors with symmetries.**

Manuscript received June 01, 2015

Virendra Nath Pathak, Department of Mathematics, Shri Ramswaroop Memorial Group of Professional Colleges, Tewariganj, Faizabad Road, Lucknow

Geeta Verma, Department of Mathematics, Shri Ramswaroop Memorial Group of Professional Colleges, Tewariganj, Faizabad Road, Lucknow

I. INTRODUCTION

PRELIMINARIES

Let V be a (2n+1)- dimensional real vector space with almost r-contact metric structure (ϕ,ξ,η,g), where ϕ is a tensor of type (1,1), ξ is a vector, η is a convector and g is a definite metric so that

$$\phi^2 x = -x + \eta^p \xi_p, \quad \phi(\xi_p) = 0, \qquad \eta^p \quad \phi = 0,$$

$$g(\xi_p,\xi_p) = 1, \quad g(\phi x,\phi y) = g(x,y) - \eta^p(x)\eta^p(y)$$

For arbitrary vectors x y in V. For arbitrary $x \in V$, we denote $hx = \phi^2 x.$

We consider the subspace $\Im \ of \ V^* \otimes V^* \otimes V^*$ defined by the conditions :

$$\Im = \{F \in \Im / F(x,y,z) = -F(x,y,z) = -F(x,\phi y,\phi z) + \eta^p(y)F(x,\xi,z) + \eta^p(z)F(x,y,\xi)\}$$

for x, y , z in V.

Let $\{e_i\}, i = 1,.........,2n+1$ be an orthonormal basis of V. The metric g induces an inner product in the vector space f :

$$< F', F'' > = \sum_{i,j,k=1}^{2n+1} F'(e_i,e_j,e_k)F''(e_i,e_j,e_k); \qquad F',\, F'' \in \Im.$$

We associate with every $F \in \Im$ the following convectors:

(2) $$f(F)(z) = \sum_i F(e_i,e_j,z), \quad f^*(F)(z) = \sum_i F(e_i,\phi e_i,z), \quad \omega(F)(z) = F(\xi_p,\xi_p,z)$$

The standard representation of $U(n)\times 1$ in V induces an associated representation of $U(n)\times 1$ in $\Im$ It is well known the following :

Lemma 1. Let L be an involutive isometry of $\Im$, which commutes with the action $U(n)\times 1$ in $\Im$. Then

$$\Im = L^+(\Im) \oplus L^-(\Im),$$

Where $L^+(\Im)$ and $L^-(\Im)$ are the eigen spaces of L corresponding to the eigen values + 1 and -1 of L. The decomposition is orthogonal and invariant under the action of $U(n)\times 1$. The components of an element $F \in \Im$ in $L^+(\Im)$ and $L^-(\Im)$ are

$$F^+ = \frac{1}{2}(F + LF), \quad F^- = \frac{1}{2}(L - LF).$$

2. ASSOCIATED FORMS WITH AN ELEMENT OF $\Im$

With every F in $\Im$ we associate the following basic forms :

$$F_1(F)(x,y,z) = \eta^p(x)F(\xi_p,y,z),$$

$$F_2(F)(x,y,z) = \eta^p(y)F(x,\xi_p,z) - \eta^p(z)F(x,\xi_p,y),$$

$$F_3(F)(x,y,z) = \eta^p(x)\eta^p(y)F(\xi_p,\xi_p,z) - \eta^p(x)\eta^p(z)F(\xi_p,\xi_p,y)$$

$$= \eta^p(x)\eta^p(y)\omega(F)(z) - \eta^p(x)\eta^p(z)\omega(F)(y)$$

$$hF(x,y,z) = F(hx,hy,hz).$$

Lemma 2. Let $F \in \Im$. Then $F_i(F)$ $(i = 1,2,3)$ and hF are elements of $\Im$ and

$$F = hF + F_1(F) + F_2(F) - F_3(F)$$

Further we consider the forms

$$F_4(F)(x,y,z) = \eta^p(y)F(\phi x,\xi_p,\phi z) - \eta^p(z)F(\phi x,\xi_p,\phi y),$$

$$F_5(F)(x,y,z) = \eta^p(y)F(z,\xi_p,x) - \eta^p(z)F(y,\xi_p,x),$$

$$F_6(F)(x,y,z) = \eta^p(y)F(\phi z,\xi_p,\phi x) - \eta^p(z)F(\phi y,\xi_p,\phi x),$$

$$F_7(F)(x,y,z) = \frac{1}{2n} f(F)(\xi_p)\{\eta^p(z)g(x,y) - \eta^p(y)g(x,z)\},$$

$$F_8(F)(x,y,z) = \frac{-1}{2n} f^*(F)(\xi_p)\{\eta^p(z)g(x,\phi y) - \eta^p(y)g(x,\phi z)\}$$

Associated with an arbitrary $F \in \Im$.

Lemma 3. Let $F \in \Im$. Then $F_i(F)$ $(i = 4,5,6,7,8)$ are elements of $\Im$.

Lemma 4. Let $F \in \Im$. The following relations are valid.

$$F_{11}(F) = F_1(F),\ F_{12}(F) = F_3(F),\ F_{13}(F) = F_3(F),\ F_{14}(F) = 0,\ F_{15}(F) = 0,$$

$$F_{21}(F) = F_3(F),\ F_{22}(F) = F_2(F),\ F_{23}(F) = F_3(F),\ F_{24}(F) = F_4(F),\ F_{25}(F) = F_5(F),$$

$$F_{31}(F) = F_3(F),\ F_{32}(F) = F_3(F),\ F_{33}(F) = F_3(F),\ F_{34}(F) = 0,\ F_{35}(F) = 0,$$

$$F_{41}(F) = 0,\ F_{42}(F) = F_4(F), F_{43}(F) = 0,\ F_{44}(F) = F_2(F) - F_3(F),\ F_{45}(F) = F_6(F),$$

$$F_{51}(F) = 0, F_{52}(F) = F_5(F),\ F_{53}(F) = 0, F_{54}(F) = F_6(F),\ F_{55}(F) = F_2(F) - F_3(F)$$

$$F_{71}(F) = F_{17}(F) = F_{73}(F) = F_{37}(F) = 0, F_{7i}(F) = F_{i7}(F) = F_7(F), i = 2,4,5,7,$$

$$F_{81}(F) = F_{18}(F) = F_{83}(F) = 0,\ F_{58}(F) = F_{85}(F) = -F_{48}(F) = -F_{84}(F) = F_{88}(F) = F_8(F),$$

$$h(F_i(F) = F_i(hF) = 0,\ i = 1,..............,8,$$

Where $F_{i,j}(F) = F_i(F_j(F))$.

Lemma 5. Let $F \in \Im$. Then we have

$$f(F_1(F)) = \omega(F),\ f^*(F_1(F)) = 0,\ \omega(F_1(F)) = \omega(F)$$

$$f(F_2(F)) = \omega(F) + f(F)(\xi_p)\eta^p,\ f^*(F_1(F)) = f^*(F)(\xi_p)\eta^p,\ \omega(F_2(F)) = \omega(F),$$

$$f(F_3(F)) = \omega(F), f^*(F_3(F)) = 0,\ \omega(F_3(F)) = \omega(F),$$

$$f(F_4(F)) = f(F)(\xi_p)\eta^p,\ f^*(F_4(F)) = f^*(F)(\xi_p)\eta^p, \omega(F_4(F)) = 0,$$

$$f(F_5(F)) = f(F)(\xi_p)\eta^p,\ f^*(F_5(F)) = -f^*(F)(\xi_p)\eta^p, \omega(F_5)(F) = 0,$$

$$f(F_6(F)) = f(F)(\xi_p)\eta^p, f^*(F_6(F)) = -f^*(F)(\xi_p)\eta^p,\ \omega(F_6(F)) = 0,$$

$$f(F_7(F)) = f(F)(\xi_p)\eta^p,\ f^*(F_7(F)) = 0,\ \omega(F_7(F)) = 0,$$

$$f(F_8(F)) = 0,\ f^*(F_8(F)) = f^*(F)(\xi_p)\eta^p,\ \omega(F_8(F)) = 0.$$

3. THE SUBSPACES $h\Im$, $v\Im$ AND $\Im_1$ OF $\Im$

The first operator L_1. Let $F \in \Im : L_1(F) = F - 2F_3(F)$.

By straight forward computations, using Lemmas 4 and 5, we obtain

Lemma 6. L_1 is an involutive isometry of $\Im$ and commutes with the action of $U(n)\times 1$. This Lemma and Lemma 1 imply immediately.

Lemma 7. $\Im_1 \oplus \Im_1^{\perp}$, where

$$\Im_1 = L_1^-(\Im) = \{F \in \Im / F = F_3(F)\},$$

$$\Im_1^{\perp} = L_1^+(\Im) = \{F \in \Im / \omega(F) = 0\}.$$

The second operator L_2. Let $F \in \Im_1^{\perp} : L_2(F) = F - 2\{F_1(F) + F_2(F)\}$.

Analogously to Lemma 6 we oblain.

Lemma 8. L_2 is an involutive isometry of $\Im_1^{\perp}$ and commutes with the action of $U(n)\times 1$. We have

$\Im_1^{\perp} = v\Im \oplus h\Im$ (orthogonally),

Where
$$v\Im = L_2^{-}(\Im_1^{\perp}) = \{F \in \Im / hF = 0,\ \omega(F) = 0\},$$
$$h\Im = L_2^{+}(\Im_1^{\perp}) = \{F \in \Im / F_1(F) = F_2(F) = 0\}.$$

Taking into account Lemmas 6, 7 and 8, we obtain a partial decomposition:

Proposition 1. $\Im = \Im_1 \oplus v\Im \oplus h\Im$. The decomposition is orthogonal and invariant under the action of $U(n)\times 1$. The corresponding components of $F \in \Im$ are

$$p_1(F) = F_3(F),\ vF = F_1(F) + F_2(F) - 2F_3(F),\ hF.$$

4. THE SUBSPACE $v\Im$ OF $\Im$

The operator L_3. Let $F \in v\Im : L_3(F) = F_2(F) - F_1(F)$.

Lemma 9. L_3 is an involutive isometry of $v\Im$ and commutes with the action of $U(n)\times 1$. We have $v\Im = \Im_8 \oplus (v\Im)'$, where

$$\Im_8 = L_3^{-}(v\Im) = \{F \in \Im / hF = 0.\ F(x, y, \xi_p) = 0\},$$
$$(v\Im)' = \Im_8^{\perp} = L_3^{+}(v\Im) = \{F \in \Im / hF = 0,\ F(\xi_p, y, z) = 0\}.$$

The corresponding components of $F \in v\Im$ are $F \in v\Im$ are $F_1(F)$ and $F_2(F)$.

The operator L_4. Let $F \in \Im_8^{\perp} = (v\Im)' : L_4(F) = -F_4(F).$

Lemma 10. L_4 is an involutive isometry of $(v\Im)' = \Im_8^{\perp}$ and commutes with the action of $U(n)\times 1$. We have

$$(v\Im)' = \Im_8^{\perp} = N\Im_8^{\perp} \oplus \tilde{N}\Im_8^{\perp},$$

Where
$$N\Im_8^{\perp} = L_4^{-}(\Im_8^{\perp}) = \{F \in \Im / F = F_4(F)\},$$
$$\tilde{N}\Im_8^{\perp} = L_4^{+}(\Im_8^{\perp}) = \{F \in \Im / F = -F_4(F)\}.$$

The corresponding components of $F \in (v\Im)' = \Im_8^{\perp}$ are

$$\frac{1}{2}\{F_2(F) + F_4(F)\},\quad \frac{1}{2}\{F_2(F) - F_4(F)\}$$

The operator L_5. Let $F \in N\ \Im_8^{\perp}\ (F \in \tilde{N}\Im_8^{\perp}) : L_5(F) = -F_5(F).$

Lemma 11. L_5 is an involutive isometry of $N\Im_8^{\perp}\ (\tilde{N}\Im_8^{\perp})$ and commutes with the action of $U(n)\times 1$. We have

$$N\Im_8^{\perp} = QS\Im \oplus QK\Im,\ \tilde{N}\Im_8^{\perp} = \Im_6 \oplus \Im_7,$$

Where

$$QS\Im = L_5^{-}(N\Im_8^{\perp}) = \{F \in \Im / F = F_4(F) = F_5(F)\},$$
$$QK\Im = L_5^{+}(N\Im_8^{\perp}) = \{F \in \Im / F = F_4(F) = -F_5(F)\},$$
$$\Im_6 = L_5^{-}(\tilde{N}\Im_8^{\perp}) = \{F \in \Im / F = -F_4(F) = F_5(F)\},$$
$$\Im_7 = L_5^{+}(\tilde{N}\Im_8^{\perp}) = \{F \in \Im / F = -F_4(F) = -F_5(F)\}.$$

The corresponding components of $F \in N\Im_8^{\perp}(F \in \tilde{N}\Im_8^{\perp})$ are

$$\frac{1}{4}\{F_2(F) + F_4(F) + F_5(F) + F_6(F)\},\quad \frac{1}{4}\{F_2(F) + F_4(F) - F_5(F) - F_6(F)\}$$
$$\left(\frac{1}{4}\{F_2(F) - F_4(F) + F_5(F) - F_6(F)\},\quad \frac{1}{4}\{F_2(F) - F_4(F) - F_5(F) + F_6(F)\}\right)$$

The operator L_6. Let $F \in QS\Im : L_6(F) = F - 2F_7(F).$

Lemma 12. L_6 is an involutive isometry of $QS\Im$ and commutes with the action of $U(n)\times 1$. We have

$$QS\Im = \Im_2 \oplus \Im_4 \text{ (Orthogonally)},$$

Where $\Im_2 = L_6^-(QS\Im) = \{F \in \Im / F = F_7(F)\}$,

$$\Im_4 = L_6^+(QS\Im) = \{F \in \Im / F = F_4(F) = F_5(F),\ f(F)(\xi_p) = 0\}.$$

The corresponding components of $F \in QS\Im$ in $\Im_2$ and $\Im_4$ are

$$F_7(F),\ \frac{1}{4}\{ F_2(F) + F_4(F) + F_5(F) + F_6(F) - 4F_7(F) \}.$$

The operator L_7. Let $F \in QK\Im : L_7(F) = F - 2F_8(F)$.

Lemma 13. L_7 is an involutive isometry of $QK\Im$ and commutes with the action of $U(n)\times 1$. We have

$$QK\Im = \Im_3 \oplus F_5 \text{ (Orthogonally)},$$

Where $\Im_3 = L_7^-(QK\Im) = \{F \in \Im / F = F_8(F)\}$,

$$\Im_5 = L_7^+(QK\Im) = \{F \in \Im / F = F_4(F) = -F_5(F),\ f^*(F)(\xi_p) = 0\}$$

The corresponding components of $F \in QK\Im$ in $\Im_3$ and $\Im_5$ are

$$F_8(F),\ \frac{1}{4}\{ F_2(F) + F_4(F) - F_5(F) - F_6(F) - 4F_8(F) \}.$$

Using lemma 9 – 13, we get

Proposition 2. $vF = \Im_2 \oplus \ldots\ldots\ldots \oplus \Im_8$. The decomposition is orthogonal and invariant under the action of $U(n)\times 1$. The corresponding components of $F \in \Im$ in $\Im_i (i = 2,\ldots\ldots\ldots,8)$ are

$$p_2(F) = F_7(F),$$
$$p_3(F) = F_8(F),$$
$$p_4(F) = \frac{1}{4}\{F_2(F) + F_4(F) + F_5(F) + F_6(F) - 4F_7(F) - F_3(F)\},$$
$$p_5(F) = \frac{1}{4}\{F_2(F) + F_4(F) + F_5(F) - F_6(F) - 4F_8(F) - F_3(F)\},$$
$$p_6(F) = \frac{1}{4}\{ F_2(F) - F_4(F) + F_5(F) - F_6(F) - F_3(F)\},$$
$$p_7(F) = \frac{1}{4}\{F_2(F) - F_4(F) - F_5(F) + F_6(F) - F_3(F)\},$$
$$p_8(F) = F_1(F) - F_3(F).$$

5. THE SUBSPACE $h\Im$

Now, let $hV = \{x \in V / x = hx\}$. Denoting the restrictions of g and ϕ on hV with the same letteres, we obtain the Hermitian vector space $\{hV, g, \phi\}$ of dimension 2n. We identify the elements of $h\Im$ with their restrictions on hV. Then we can consider the vector space $h\Im$ as the vector space of the tensors hF of type (0, 3) over hV having the properties

$$hF(x, y, z) = -hF(x, z, y) = -hF(x, \phi y, \phi z)$$

For all $x, y, z \in hV$. The action $U(n)\times 1$ on $h\Im$ coincide with the action of $U(n)$ on $h\Im$. In [1] the vector space $h\Im$ has been decomposed orthogonally into irreducible components invariant under the action $U(n)$.

Let $F \in h\Im$. It is not difficult to verify that the forms.

$$F_9(F) = \frac{1}{2(n-1)}\left\{\begin{matrix} g(hx, hy)f(F)(z) - g(hx, hz)f(F)(y) - g(x, \phi y)f(F)(\phi z) \\ + g(x, \phi z)f(F)(\phi y) \end{matrix}\right\}$$

$$F_{10}(F)=\frac{1}{2}\{F(x,y,z)+f(\phi x,\phi y,z)\}$$

$$F_{11}(F)=\frac{1}{6}\{F(x,y,z)+F(y,z,x)+F(z,x,y)-F(\phi x,\phi y,z)-F(\phi y,\phi z,x)-F(\phi z,\phi x,y)\}$$

$$F_{12}=\frac{1}{2}\{F(x,y,z)-F(\phi x,\phi y,z)\}$$

Are also elements of $h\Im$.

Using the decomposition in [1] we have

Proposition 3. $h\Im=F_9\oplus F_{10}\oplus F_{11}\oplus F_{12}$, where

$$\Im_9=\{F\in\Im / F=hF=F_9(F)\},$$
$$\Im_{10}=\{F\in\Im / F=hF=F_{10}(F)-F_9(F)\},$$
$$\Im_{11}=\{F\in\Im / F=hF=F_{11}(F)\},$$
$$\Im_{12}=\{F\in\Im . F=hF=F_{12}(F)-F_{11}(F)\}.$$

The decomposition is orthogonal and invariant under the action of $U(n)\times 1$. The corresponding components of $F\in\Im$ are

$$F_9(F),\ F_{10}(F)-F_9(F),\quad F_{11}(F),\ F_{12}(F)-F_{11}(F).$$

6. APPLICATIONS TO ALMOST r- CONTACT METRIC MANIFOLDS

Let M be an almost r-contact metric manifold with structure (ϕ,ξ_p,η^p,g), where ϕ is a tensor field of type (1,1) ξ_p is a tensor field , η^p is a 1-form , and g is a Riemannian metric on M such that

$$\phi^2 x=-x+\eta^p(x)\xi_p,\quad g(\xi_p,\xi_p)=1,\quad \eta^p\quad \phi=0$$
$$\phi\xi_p=0,\ g(\phi x,\phi y)=g(x,y)-\eta^p(x)\eta^p(y).$$

For arbitrary vector fields x,y on M. For all vector fields x,y on M we denote (3)

$$F(x,y,z)=g((\nabla_x\phi)y,z).$$

Let T_PM be the tangent space to M at $p\in M$ and $V=T_pM$. The restriction F_p of F given by (3) on V has the properties (1). We shall call M is of class $W_i(i=1,......,12)$ if F_p is in the subspace $\Im_i(i=1,.......,12)$ for every $p\in M$. Using the propositions 1, 2 and 3 we obtain 12 basis classes of almost r- contact metric manifolds. Further we give the defining conditions for these classes. Let F be given by (3) and $f,\ f^*,\omega$ be $f(F),\ f^*(F),\omega$ respectively defined by (2)

The class W_1 :

$$F(x,y,z)=\eta^p(x)\eta^p(y)\omega(z)-\eta^p(x)\eta^p(z)\omega(y).$$

The class W_2 :

$$F(x,y,z)=\frac{f(\xi_p)}{2n}\{\eta^p(z)g(x,y)-\eta^p(y)g(x,z)\}.$$

This is the class of $\alpha-r-$Sasakian manifolds.

The class W_3 :

$$F(x,y,z)=-\frac{f^*(\xi_p)}{2n}\{\eta^p(z)g(x,\phi y)-\eta^p(y)g(x,\phi z)\}.$$

This is the class of $\alpha-r-$Kenmotsu manifolds.

The class W_4 :

$$F(x,y,z)=\eta^{p}(y)F(\phi X,\xi_{P},\phi z)-\eta^{p}(z)F(\phi x,\xi_{p},\phi y)$$
$$=\eta^{p}(y)F(z,\xi_{p},x)-\eta^{p}(z)F(y,\xi_{p},x), \quad f(\xi_{p})=0$$

The class W_5 :

$$F(x,y,z)=\eta^{p}(y)F(\phi x,\xi_{p},\phi z)-\eta^{p}(z)F(\phi x,\xi_{p},\phi y)$$
$$=-\eta^{p}(y)F(z,\xi_{p},x)+\eta^{p}(z)F(y,\xi_{p},x), \qquad f^{*}(\xi_{p})=0$$

The class W_6 :

$$F(x,y,z)=-\eta^{p}(y)F(\phi x,\xi_{p},\phi z)+\eta^{p}(z)F(\phi x,\xi_{p},\phi y)$$
$$=\eta^{p}(y)F(z,\xi_{p},x)-\eta^{p}(z)F(y,\xi_{p},x)$$

The class W_7 :

$$F(x,y,z)=-\eta^{p}(y)F(\phi x,\xi_{p},\phi z)+\eta^{p}(z)F(\phi x,\xi_{p},\phi y)$$
$$=-\eta^{p}(y)F(z,\xi_{p},x)+\eta^{p}(z)F(y,\xi_{p},x)$$

The class W_8 :

$$F(hx,hy,hz)=F(x,y,\xi_{p})=0$$

The class W_9 :

$$F(\xi_{p},y,z)=F(x,y,\xi_{p})=0$$
$$F(x,y,z)=\frac{1}{2(n-1)}[\{g(\phi x,\phi y)f(z)-g(\phi x,\phi z)f(y)\}-g(x,\phi y)f(\phi z)+g(x,\phi z)f(\phi y)]$$

The class W_{10} :

$$F(\xi_{p},y,z)=F(x,y,\xi_{p})=0$$
$$F(\phi x,\phi y,z)-F(x,y,z)=0, \quad f=0$$

The class W_{11} :

$$F(\xi_{p},y,z)=F(x,y,\xi_{p})=0$$
$$F(x,x,z)=0$$

The class W_{12} :

$$F(\xi_{p},y,z)=F(x,y,\xi_{p})=0$$
$$F(x,y,z)+F(y,z,x)+F(z,x,y)=0$$

The class of cosymplectic manifolds is characterized by $F=0$. This class is contained in all $W_i (i=1,2,......,12)$. An almost r- contact metric manifold M belongs to two classes $W_i, W_j (i\neq j)$ iff M is cosymplectic.

REFERENCES

1. Gray A. and L. Hervella. The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants. Ann. Mat. Pura. Appl. 123 (1980), 35-58.
2. Blair, D. E. : Contact Manifolds in Riemannian Geometry , Lecture Notes in Maths., 509, Springer-Verlag, 1976.
3. Arslan, K., Murathan, C. and Yilidiz. A. : Contant Metric R-harmonic manifold, Balkan J. of Geom. Its Appl., 5 (2000), 1-6.
4. Shaikh, Abbas Ali and Hui, Shyamal Kumar : On extended generalized ϕ -recurrent Sasakian Manifolds, Appliesd Mathematics, 2 (2011), 1317-1322.

De, U. C., Yildiz, A. and Yaliniz, A. F. : On ϕ -recurrent Kenmotsu Manifolds, Turk J. Math., 33(2009), 17-25.