Numerical Solution of the Poisson Equation in Nonrectangular Domains

Dr.Cüneyt Ezgi, Dr.İbrahim Girgin, Ömer Faruk Çakmak

***Abstract*— In this study, finite difference numerical solution of the Poisson equation in nonrectangular domains is introduced and a 2D case study in nonrectangular domain is presented. The problem is solved both manually and numerically with Matlab. Both results are compared. The temperature distribution solution of Poisson equation is presented in 3-D and contour plots**

***Index Terms*— Nonrectangular domain, Numerical solution, Poisson equation, Temperature distribution**

I. Introduction

The Laplacian equation in Cartesian coordinates is given as

$$\nabla^2 u = f(x, y, z) \qquad (1)$$

There are two types of Laplacian equations: Homogeneous and Non-homogeneous.

The Homogeneous Laplace equation is given as

$$\nabla^2 u = 0 \qquad (2)$$

One of the applications of Laplace equation is in heat transfer: Temperature distribution in a plane with constant thermal conductivity, no heat generation, and steady state case. For this case Laplace equation is given as

$$\nabla^2 T = 0 \qquad (3)$$

which can be written for the three dimensional temperature distribution as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (4)$$

and for the two dimensional temperature distribution for the Cartesian coordinate system.

$$\nabla^2 u = f \qquad (5)$$

Manuscript received August 10, 2015.

Cüneyt Ezgi, Mechanical Engineering Department, Turkish Naval Academy, Tuzla, Istanbul, Turkey
İbrahim Girgin, Mechanical Engineering Department, Turkish Naval Academy, Tuzla, Istanbul, Turkey
Ömer Faruk Çakmak, Mechanical Engineering Department, Turkish Naval Academy, Tuzla, Istanbul, Turkey

where f is a source function.

For the two dimensional case,

$$\nabla^2 u = u_{xx} + u_{yy} = f(x, y) \qquad (6)$$

f is the heat generation term in heat transfer. For this case, the equation can be written as (Çengel and Ghajar, 2011)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{e}{k} = 0 \qquad (7)$$

where e is the rate of heat generated in the unit volume, and k is the thermal conductivity. $\frac{e}{k}$ is the source term, f in non-homogeneous Laplace equation. For the steady state two dimensional case with heat generation and constant thermal conductivity, this governing equation for the temperature distribution can also be expressed as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{e}{k} = 0 \qquad (8)$$

If $\Delta x = \Delta y \equiv h$, finite-difference approximation to the Poisson equation

$$U_{j-1,k} + U_{j,k-1} + U_{j+1,k} + U_{j,k+1} - 4U_{j,k} = h^2 f_{j,k} \qquad (9)$$

We can not apply the finite difference scheme (9) at grid points such as P because the points N and E do not fall on the boundary curve C, as illustrated in Figure 1. (Greenberg, 1988; Kreyszig, 1998)

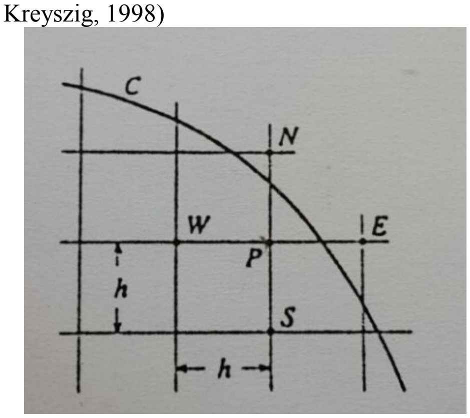


Figure 1. Nonrectangular domain

We slide N and E so that they do fall on C, as shown Fig. 2. (Greenberg, 1988; Kreyszig, 1998)

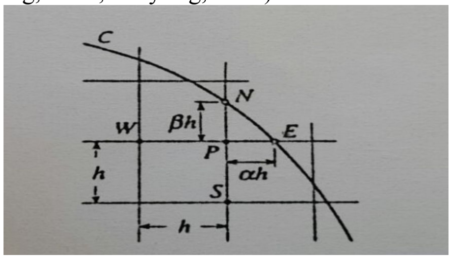


Figure 2. Sliding of N and E

The general case is shown in Fig. 3, where

0<α≤1, 0<β≤1, 0<γ≤1, 0<δ≤1.

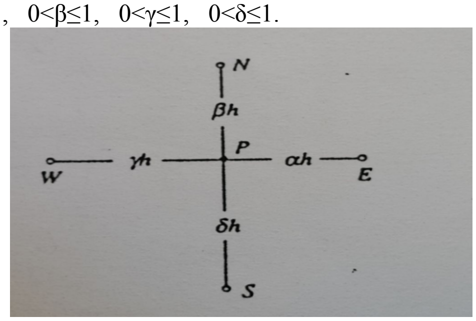


Figure 3. General case

Taylor expansions about P (the point x_j, y_k) in the eastern and western directions, respectively are as follows (Greenberg, 1988; Kreyszig, 1998)

$$u(x_j+\alpha h, y_k)=u(x_j,y_k)+u_x(x_j,y_k)\alpha h + \frac{1}{2!}u_{xx}(x_{j,y_k})(\alpha h)^2+\ldots, \quad (10)$$

$$u(x_j-\gamma h, y_k)=u(x_j,y_k)+u_x(x_j,y_k)(-\gamma h) + \frac{1}{2!}u_{xx}(x_j,y_k)(-\delta h)^2+\ldots \quad (11)$$

or using N, E, S,W, P subscript notation instead,

$$u_E=u_P+u_x\big|_P\alpha h+\frac{1}{2}u_{xx}\big|_P\alpha^2h^2+\ldots, \quad (12)$$

$$u_W=u_P-u_x\big|_P\gamma h+\frac{1}{2}u_{xx}\big|_P\gamma^2h^2-\ldots \quad (13)$$

Multiplying (12) by γ and (13) by α and adding, to cancel the u_x terms, gives

$$\gamma u_E+\alpha u_W=(\gamma+\alpha)u_P+\frac{1}{2}(\alpha^2\gamma+\alpha\gamma^2)h^2u_{xx}\big|_P+\ldots \quad (14)$$

If we neglect terms of order h^3 and higher in (14) then we obtain

$$u_{xx}\big|_P\approx\frac{2}{\alpha\gamma(\alpha+\gamma)h^2}\left[\gamma u_E+\alpha u_W-(\gamma+\alpha)u_P\right] \quad (15)$$

Taylor expansions about P in the northern and southern directions give the result

$$u_{yy}\big|_P\approx\frac{2}{\beta\delta(\beta+\gamma)h^2}\left[\delta u_N+\beta u_S-(\delta+\beta)u_P\right] \quad (16)$$

Using (15, 16), finite-difference approximation to the Poisson equation $u_{xx}+u_{yy}=f(x,y)$, at P, becomes (Greenberg, 1988; Kreyszig, 1998)

$$\frac{2}{\gamma(\gamma+\alpha)}U_W+\frac{2}{\delta(\delta+\beta)}U_S+\frac{2}{\alpha(\alpha+\gamma)}U_E+\frac{2}{\beta(\beta+\delta)}U_N-2\frac{\alpha\gamma+\beta\delta}{\alpha\beta\gamma\delta}U_P=h^2f_P \quad (17)$$

If $\alpha=\beta=\gamma=\delta=1$, then (17) reduces to

$$U_{j-1,k}+U_{j,k-1}+U_{j+1,k}+U_{j,k+1}-4U_{j,k}=h^2f_{j,k} \quad (18)$$

II. A CASE STUDY

Let us consider the problem illustrated in Figure 4.

$u_{xx}+u_{yy}=f(x,y)$
with $u_1=u_2=u_3=u_4=0$ and $u_5=50$ and $f(x,y)=-20$

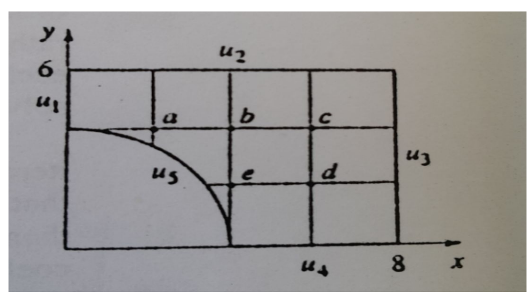


Figure 4. Case study

so we have

At a,
αh=2, α=2/2=1
βh=2, β=2/2=1
γh=2, γ=2/2=1
δh=4-$\sqrt{4^2-2^2}$ =0.54, δ=0.54/2=0.27

At b,

αh=2, α=2/2=1
βh=2, β=2/2=1
γh=2, γ=2/2=1
δh=2, δ=2/2=1

At c,
αh=2, α=2/2=1
βh=2, β=2/2=1
γh=2, γ=2/2=1
δh=2, δ=2/2=1

At d,

αh=2, α=2/2=1
βh=2, β=2/2=1
γh=2, γ=2/2=1
δh=2, δ=2/2=1

At e,
αh=2, α=2/2=1
βh=2, β=2/2=1
γh=4- $\sqrt{4^2-2^2}$ =0.54, γ=0.54/2=0.27
δh=2, δ=2/2=1

Thus, writing (17) at these points gives the equations

For a,

We use the average value for western direction,

$$U_W=\frac{0+50}{2}=25$$

$$\frac{2}{1(1+1)}25+\frac{2}{0.27(0.27+1)}50+\frac{2}{1(1+1)}U_b+\frac{2}{1(1+0.27)}0-2\frac{1x1+1x0.27}{1x1x1x0.27}U_a=-80$$

For b,

$$\frac{2}{1(1+1)}U_a+\frac{2}{1(1+1)}U_e+\frac{2}{1(1+1)}U_c+\frac{2}{1(1+1)}0-2\frac{1x1+1x1}{1x1x1x1}U_b=-80$$

For c,

$$\frac{2}{1(1+1)}U_b+\frac{2}{1(1+1)}U_d+\frac{2}{1(1+1)}0+\frac{2}{1(1+0.27)}0-2\frac{1x1+1x1}{1x1x1x1}U_c=-80$$

For d,

$$\frac{2}{1(1+1)}U_e+\frac{2}{1(1+1)}0+\frac{2}{1(1+1)}0+\frac{2}{1(1+1)}U_c-2\frac{1x1+1x1}{1x1x1x1}U_d=-80$$

For e,

Similarly, for southern direction

$$U_S=\frac{0+50}{2}=25$$

$$\frac{2}{0.27(0.27+1)}50+\frac{2}{1(1+1)}25+\frac{2}{1(1+0.27)}U_d+\frac{2}{1(1+1)}U_b-2\frac{1x1+1x1}{1x1x0.27x1}U_e=-80$$

Then we obtain

$$\begin{bmatrix}-9.40 & 1 & 0 & 0 & 0\\ 1 & -4 & 1 & 0 & 1\\ 0 & 1 & -4 & 1 & 0\\ 0 & 0 & 1 & -4 & 1\\ 0 & 1 & 0 & 1.57 & -9.40\end{bmatrix}\begin{bmatrix}U_a\\U_b\\U_c\\U_d\\U_e\end{bmatrix}=\begin{bmatrix}-396\\-80\\-80\\-80\\-396\end{bmatrix}$$

$$\begin{bmatrix}U_a\\U_b\\U_c\\U_d\\U_e\end{bmatrix}=\begin{bmatrix}48.2383\\57.4404\\45.7053\\45.3808\\55.8179\end{bmatrix}$$

III. THE SOLUTION WITH MATLAB

The problem has been solved with Matlab using a convergence criterion of 1e-7. The differential equation is the governing equation for 2-D steady state temperature distribution in a plate with constant thermal conductivity. The temperature distribution has been solved for the distance between grid points (h) of 2 and 0.05. Then, we obtain

$$\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 25.0000 & 48.2654 & 57.4509 & 45.7089 & 0\\ NaN & NaN & 55.8295 & 45.3846 & 0\\ NaN & NaN & 25.0000 & 0 & 0\end{bmatrix}$$

It is seen that the results are very close to the manual calculations of Figure 5, which is presented above.

The temperature distribution has also been solved for the h of 0.05. The numerical mesh and the temperature distribution is seen in Figure 5. The quarter circle side has a temperature of 50, while all the other sides are taken to be zero. The average temperature of 25 has been taken as the temperatures of the corners that are the intersection of the sides and the curved line.

The temperature distribution of the plane is presented in figures 5 and 6. It is seen that the temperature at the center is much more than the maximum boundary condition temperature, 50^{o}C because of the heat generation inside the geometry. The temperature contours on the plate are illustrated in Fig. 7.

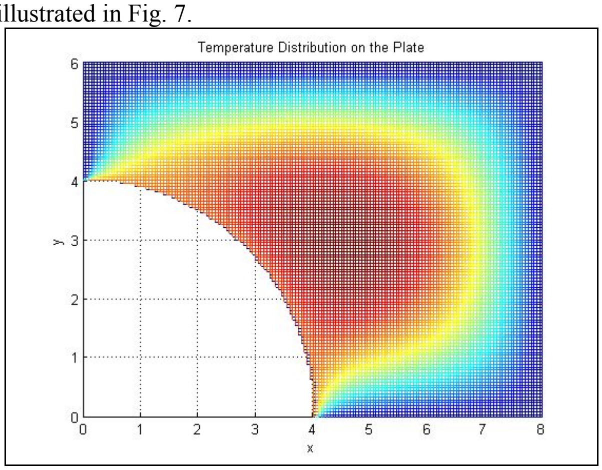


Figure 5. The numerical mesh and the temperature distribution.

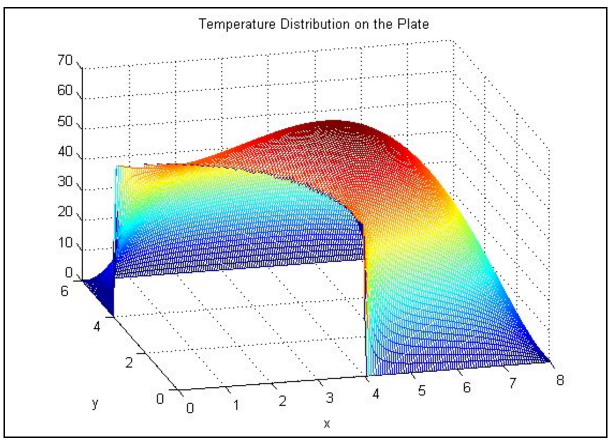


Figure 6. The temperature distribution of the plane.

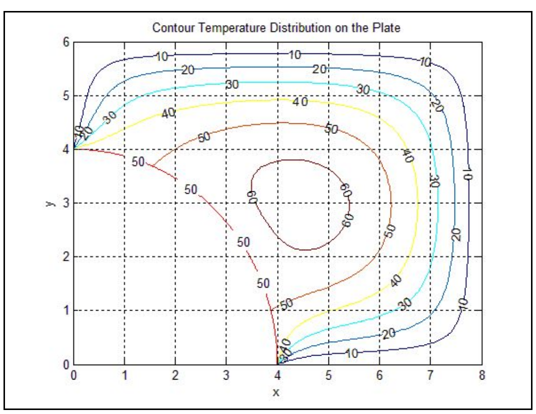


Figure 7. The temperature contours on the plate.

IV. RESULT

In this study, numerical solution of the Poisson equation in nonrectangular domains is introduced and a 2D case study in nonrectangular domain is presented. The problem is solved manually and numerically with Matlab using finite difference approach, for the temperature distribution of a plane, with heat source. The distance between the grid points for the curved side has been considered as a variable. Therefore, the accuracy of the solution has been increased.

NOMENCLATURE

e rate of heat generated in the unit volume [W/m^3]

k thermal conductivity [W/mK]

$\frac{e}{k}$ source term [K/m^2]

f source function

T temperature [K]

REFERENCES

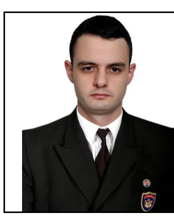
Çengel, A. Yunus, Ghajar, J., Afshın, 2011, Heat and Mass Transfer, McGraw Hill, 2011.

Greenberg, M.D.1998, Advanced Engineering Mathematics (2nd ed.): Prentice Hall.

Kreyszig, E., 1998, Advanced Engineering Mathematics (8th ed.): Wiley

Cüneyt Ezgi is an Assistant Professor in the Department of Mechanical Engineering at Turkish Naval Academy. He graduated from the Mechanical Engineering in Gazi University in 1991. Dr. Ezgi received his Msc and Ph.D. degrees in the Department of Mechanical Engineering in Ege University in 2004 and 2009. His general research interests are heat transfer, thermodynamics, thermal systems engineering, heat exchangers, heat pumps, renewable energy technologies, fuel cells.

İbrahim Girgin is an Assistant Professor in the Department of Mechanical Engineering at Turkish Naval Academy. He graduated from the Mechanical Engineering in Turkish Naval Academy, in 1992. Dr. Girgin received his Msc degree from Naval Postgraduate School in the USA in 1998, and Ph.D. degree in Istanbul Technical University in 2007. His general research interests are heat transfer, thermodynamics, thermal systems engineering, fluid mechanics, numerical methods.



Ömer Faruk Çakmak graduated from the Mechanical Engineering at Gazi University in 2014. His general research interests are heat transfer, thermodynamics, gas turbines, steam boilers