Some Classes of almost r-Contact Riemannian Manifolds and Kenmotsu Manifold

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***Abstract*— Certain classes of almost r-contact Riemannian Manifolds, viz., almost Kenmotsu, nearly Kenmotsu, Quasi-Kenmotsu and special r-contact metric Manifolds are defined and obtained some properties of these manifolds. Also, it has been shown that the structure vector field ξ of the almost r-contact metric structure ($\Phi, \xi, \eta, G)$ is not a Killing vector field on a nearly Kenmotsu vector field**

***Index Terms*— almost r-contact Riemannian manifold, Sasakian manifold, Kenmotsu Manifold, Killing vector field.**

I. Introduction

The study of odd dimensional manifolds with r- contact and almost r-contact structures was initiated by Boothby and Wang in 1958 rather from topological point of view. Sasaki and Hatakeyama reinvestigated them using tensor calculus in1961.Almost r-contact metric structures and Sasakian structures viz., almost Sasakian , nearly Sasakian etc., were proposed by Sasaki [5] in 1960 and 1965 respectively. Later, Kenmotsu [3] defined a class of almost r-contact Riemannian manifold, called Kenmotsu manifold, similar in parallel to Sasakian manifold in 1972. In this paper, we defined almost Kenmotsu, nearly Kenmotsu, Quasi-Kenmotsu and special r-contact metric Manifolds. The relation among these manifolds has been obtained and studied some properties of these manifolds.

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1 . Peliminaries

Let $M = M^{2m+r}$ be a $(2m+r)$ dimensional almost r-contact metric manifold with structure tensors $((\Phi, \xi_p, \eta^p, G)$ where Φ is a tensor of type (1,1), ξ_p is a vector field , η^p is a 1-form and G is the associated Riemannian metric on M. Then definition, we have

(1.1)
$$\Phi^2 X = \overline{\overline{X}} = -X + \eta^p(X)\xi_p, \quad \Phi\xi_p = 0$$
$$G(\Phi X, \Phi Y) = G(X,Y) - \eta^p(X)\eta^p(Y)$$

The fundamental 2-form Ω is defined by:

(1.2) $\Omega(X,Y) = G(\overline{X}, Y)$ where we put $\overline{X} = \Phi X.$

If M is an almost r-contact metric manifold, we have

(1.3) $$(D_X\Omega)(Y, \xi_p) = -(D_X\eta^p)(\overline{Y})$$

(1.4) $$(D_X\Omega)(\overline{Y}, Z) - (Y, \overline{Z}) = (D_X\eta^p)(Y)\eta^p(Z) + (D_X\eta^p)(Z)\eta^p(Y)$$

Where D is the Riemannian connection determined by the metric G.
On the almost r-contact metric manifold if further we have

(1.5) $$(D_X\Phi)(Y) = -\eta^p(Y)(\overline{X}) - G(X, \overline{Y})\xi_p$$

It is called Kenmotsu Manifold [3].
From (1.1) and (1.5), we get

(1.6) $$D_X\xi_p = -\overline{\overline{X}} = X - \eta^p(X)\xi_p$$

Then from equations (1.1) and (1.6), we get

(1.7) $$(D_X\eta^p)(Y) = G(X,Y) - \eta^p(X)\eta^p(Y) = G(\overline{X}, \overline{Y})$$

Similarly from (1.5) we also have

(1.8)
$$(D_X\Omega)(Y,Z) + (D_Y\Omega)(Z,X) + (D_Z\Omega)(X,Y)$$
$$= -2[\eta^p(X)G(Y,\overline{Z}) + \eta^p(Y)G(Z,\overline{X}) + \eta^p(Z)G(X,\overline{Y})$$

An almost r-contact structure is said to be Normal if $N \ (X,Y)$ vanishes, where

(1.9) $$N\ (X,Y) \overset{def}{=} N_\Phi(X,Y) + d\eta^p(X,Y)\xi_p$$

Here, $N_\Phi(X,Y)$ is known as the Nijenhuis tensor of Φ .

2. Almost Kenmotsu and S-r-contact metric manifolds:

Definition (2.1): An almost r-contact metric manifold M on which there exists a function f such that $\eta^p = df$ if $d\eta^p = 0.$ Then the manifold M is called an Almost Kenmtsu manifold (or) r-contact metric manifold.
From the above definition, if D is an affine connection on r-contact metric manifold, we have

(2.1) (a) $$(D_X\eta^p)(Y) - (D_Y\eta^p)(X) + \eta^p[T(X,Y)] = 0$$

Where T is the torsion tensor of D.
If D is symmetric, then from (2.1)(a), we have

(2.1) (b) $(D_X \eta^p)(Y) - (D_Y \eta^p)(X) = 0$

Note that , in the sequel, we shall always take D as a Riemannian connection in this paper.

Definition (2.2): An almost Kenmotsu manifold (r-contact metric manifold) on which if the condition

(2.2) $(D_X \eta^p)(Y) + (D_Y \eta^p)(X) = 2G(\overline{X}, \overline{Y})$

Is satisfied, then it is called a special r-contact metric manifold or in short S-r-contact metric manifold.

Therefore, from (2.1)(b) and (2.2), for an S-r-contact metric manifold, we get

(2.3) $(D_X \eta^p)(Y) = (D_Y \eta^p)(X) = G(\overline{X}, \overline{Y}) = -\Omega(\overline{X}, Y)$

Theorem (2.1): In an almost r-contact metric manifold, if $D_X \xi_p = X - \eta^p(X)\xi_p$ is satisfied, it is an S-r-contact metric manifold.

Proof: On the almost r-contact metric manifold, we have

$$(D_X \eta^p)(Y) = G(D_X \xi_p, Y) = G(X - \eta^p(X)\xi_p, Y) = G(\overline{X}, \overline{Y})$$

Similarly, we see that

$$(D_Y \eta^p)(X) = G(\overline{Y}, \overline{X})$$

On adding and subtracting the above two values, we get both (2.2) and (2.3) respectively, which proves the theorem.

Preposition (2.1): In an S-r-contact metric manifold, we get

(2.4) $D_X \xi_p = X - \eta^p(X)\xi_p$

Proof: The equation (2.3) is equivalent to (2.4)

Corollary (2.1): The following holds on S-r-contact metric manifold:

(2.5) $(D_X \Omega)(Y, \xi_p) = G(\overline{X}, Y)$

Proof: Taking the covariant differentiation of $\Omega(Y, \xi_p) = 0,$ we get

$$(D_X \Omega)(Y, \xi_p) = -\Omega(Y, D_X \xi_p) = \Omega(Y, \overline{\overline{X}}) = G(\overline{X}, Y)$$

Theorem (2.2) : In an S-r-contact metric manifold, we have

(2.6)
$$(i) \quad (D_{\overline{Z}} \Omega)(X, Y) = -'K(X, Y, Z, \xi_p)$$
$$(ii) \quad (D_{\overline{Z}} \Phi)(X) = -K(Z, \xi_p, X)$$

Proof: The equation $-\Omega(\overline{Y}, Z) = (D_Y \eta^p)(Z)$ implies

$$-(D_X \Omega)(\overline{Y}, Z) + (D_X \Omega)(Y, \overline{Z}) + (D_Y \Omega)(\overline{X}, Z) - (D_Y \Omega)(X, \overline{Z})$$
$$= G(D_X D_Y \xi_p - D_Y D_X \xi_p - D_{[X,Y]}\xi_p, Z)$$

Using (1.5), the above equation implies

$$\eta^p(Y)G(X, Z) - \eta^p(X)G(Y, Z) = -'K(X, Y, Z, \xi_p)$$

Which can also be written a (2.6) (i) and hence, also we have (2.6) (ii).

Theorem (2.3) : On an S-r-contact metric manifold, the condition

(2.7) $(D_Z \Omega)(X, Y) = \eta^p(X)G(Y, Z) - \eta^p(Y)G(X, Z)$

Is equivalent to the condition

(2.8) $(D_{\overline{Z}} \Omega)(\overline{X}, \overline{Y}) = -'K(\overline{X}, \overline{Y}, Z, \xi_p) = 0$

Proof: From (2.6) and (2.7) we get

$$(D_{\overline{Z}} \Omega)(X, Y) = -'K(X, Y, Z, \xi_p) = 0$$
$$= \eta^p(X)G(Y, Z) - \eta^p(Y)G(X, Z)$$

Barring X and Y in the above equation, we get (2.8). Again, barring X and Y in (2.8), we get

$$(D_{\overline{Z}}\Omega)(X,Y)=\eta^{p}(Y)(D_{\overline{Z}}\Omega)(X,\xi_{p})-\eta^{p}(X)(D_{\overline{Z}}\Omega)(Y,\xi_{p})$$

Using (2.5), the above equation gives (2.7).

3. Quasi Kenmotsu manifold:

Definition (3.1): An almost r-contact metric manifold is said to be Quasi Kenmotsu manifold if

(3.1) $$(D_{\overline{X}}\Omega)(Y,Z)+(D_{\overline{Y}}\Omega)(Z,X)+(D_{\overline{Z}}\Omega)(X,Y)=0$$

Theorem (3.1): The necessary and sufficient condition for a Quasi-Kenmotsu manifold to be Normal is

(3.2) $$(D_{Z}\Omega)(X,\overline{Y})=-\eta^{p}(X)(D_{Z}\eta^{p})(Y).$$

Proof: We know that the necessary and sufficient condition for a Quasi – Kenmotsu manifold which is an almost r-contact manifold to be normal is $N \quad =0$. That is,

$$N_{\Phi}(X,Y)+d\eta^{p}(X,Y)\xi_{p}=0,$$

Which is represented as

$$(D_{\overline{X}}\Omega)(Y,Z)-(D_{\overline{Y}}\Omega)(X,Z)+(D_{X}\Omega)(Y,\overline{Z})-(D_{Y}\Omega)(X,\overline{Z})+d\eta^{p}(X,Y)\eta^{p}(Z)=0$$ By using the equations (1.8),(1.4) and (2.3) the above equation gives (3.2).

4. Kenmotsu manifold:

Definition (4.1): An S-r-contact metric manifold on which the equation (2.8) holds is called a Kenmotsu manifold.

Theorem (4.1): A normal S-r-contact metric manifold is Kenmotsu.

Proof: By $N \quad (X,Y)=0,$ we have

$$(D_{\overline{X}}\Omega)(Y,Z)-(D_{\overline{Y}}\Omega)(X,Z)+(D_{X}\Omega)(Y,\overline{Z})-(D_{Y}\Omega)(X,\overline{Z})+d\eta^{p}(X,Y)\eta^{p}(Z)=0$$

Using the equations (2.1), (1.8),(1.4) and (2.3) respectively, the above equation gives

$$(D_{Z}\Omega)(X,\overline{Y})=-\eta^{p}(X)G(Y,Z)+\eta^{p}(X)\eta^{p}(Y)\eta^{P}(Z),$$

Which also implies

$$(D_{Z}\Omega)(X,Y)=-\eta^{p}(X)G(\overline{Z},Y)+\eta^{p}(Y)G(Z,\overline{X}).$$

This shows that the manifold is of Kenmotsu type. Hence, an alternate definition of Kenmotsu manifold is given as: A Kenmotsu manifold is a Normal S-r – contact metric manifold.

5. Nearly Kenmotsu manifold:

Definition (5.1) : An almost r-contact metric manifold on which

(5.1) $$(D_{X}\Phi)(Y)+(D_{Y}\Phi)(X)=-\eta^{p}(Y)(\overline{X})-\eta^{p}(X)(\overline{Y})$$

Is satisfied, is called a nearly Kenmotsu manifold.

Theorem (5.1): On nearly Kenmotsu manifold ξ_{p} is not a Killing vector field.

Proof: Operate (5.1) with G and put $Y=\xi_{p}$, we get

$$(D_{X}\Omega)(\xi_{p},Z)+(D_{\xi_{p}}\Omega)(X,Z)=\Omega(Z,X)$$

Using (1.3) and barring Z in this equation, we see that

(5.2) $$-(D_{X}\eta^{p})(Z)+(D_{\xi_{p}}\Omega)(X,\overline{Z})=-\Omega(\overline{Z},X)$$

Consequently, we have

$$-[(D_{X}\eta^{p})(Z)+(D_{Z}\eta^{p})(X)]-[(D_{\xi_{p}}\Omega)(\overline{X},Z)-(D_{\xi_{p}}\Omega)(X,\overline{Z})]=2\Omega(\overline{X},Z)$$

Use of (1.4) in this equation yields

(5.3) $$-[(D_{X}\eta^{p})(Z)+(D_{Z}\eta^{p})(X)]-(D_{\xi_{p}}\eta^{p})(X)\eta^{p}(Z)-(D_{\xi_{p}}\eta^{p})(Z)\eta^{p}(X)=2\Omega(\overline{X},Z)$$

Writing ξ_{p} for X in (5.2), we get $(D_{\xi_{p}}\eta^{p})(Z)=0$ which proves the theorem.

Theorem (5.2): A normal nearly Kenmotsu manifold is Kenmotsu.

Proof: Since we have $N = 0$, it holds that

$$[\overline{X},\overline{Y}]+[\overline{\overline{X},\overline{Y}}]-[\overline{\overline{X},Y}]-[\overline{X,\overline{Y}}]+d\eta^{p}(X,Y)\xi_{p}=0$$

Operating the above by η^{p} implies

(5.4) $$\eta^{p}[D_{\overline{X}}\phi)Y-(D_{\overline{Y}}\phi)X]+d\eta^{p}(X,Y)=0$$

Barring X in (5.1), and on operation with η^{p}, we get

(5.5) $$\eta^{p}(D_{\overline{X}}\phi)Y=-\eta^{p}(D_{Y}\phi)\overline{X}.$$

Similarly, we see that

(5.6) $$\eta^{p}(D_{\overline{Y}}\phi)X=-\eta^{p}(D_{X}\phi)\overline{Y}$$

Using (5.5) and (5.6), equation (5.4) assumes the form

$$(D_{X}\Omega)(\overline{Y},\xi_{p})-(D_{Y}\Omega)(\overline{X},\xi_{p})+d\eta^{p}(X,Y)=0.$$

Use of (1.3) in this equation yields $d\eta^{p}(X,Y)=0$ which shows that the manifold is of almost Kenmotsu. By equation (2.2) it is of S-r-contact metric manifold and therefore the result follows from theorem (4.1).

Theorem (5.3): A manifold which is of nearly Kenmotsu and Quasi Kenmotsu is of Kenmotsu.

Proof: On a Nearly Kenmotsu manifold, we have

$$(D_{X}\Omega)(Y,Z)+(D_{Y}\Omega)(X,Z)=-\eta^{p}(X)G(\overline{Y},Z)-\eta^{p}(Y)G(\overline{X},Z)$$

Adding the above equation to (1.8), we get

(5.7) $$2(D_{X}\Omega)(Y,Z)+(D_{Z}\Omega)(X,Y)=-3\eta^{p}(Y)G(\overline{X},Z)-\eta^{p}(X)G(\overline{Z},Y)-2\eta^{p}(Z)G(\overline{Y},X)$$

From (1.5) we have

(5.8) $$(D_{X}\Omega)(Z,Y)+(D_{Z}\Omega)(X,Y)=-\eta^{p}(X)G(\overline{Z},Y)-\eta^{p}(Z)G(\overline{X},Y)$$

Hence from (5.7) and (5.8) we have

$$(D_{X}\Omega)(Y,Z)=-\eta^{p}(Y)G(\overline{X},Z)-\eta^{p}(Z)G(\overline{Y},X)$$

which proves

$$(D_{X}\Phi)(Y)=-\eta^{p}(Y)(\overline{X})-G(X,\overline{Y})\xi_{p}$$

It shows that the manifold is Kenmotsu.

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