Some Fixed Point Theorems for Weakly Commuting Mappings Related to Fuzzy-3 Metric Spaces

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Abstract— In the present paper some generalization on fixed point and common fixed point theorems in complete Fuzzy 3-metric spaces are established

Index Terms— Fuzzy metric spaces, fuzzy 3- metric spaces, fixed point, Common fixed point.

I. INTRODUCTION

Highlight a section that you want to designate with a certain style, The study of common fixed points of mappings in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. In 1965, the concept of fuzzy sets was introduced by Zadeh [36]. With the concept of fuzzy sets, the fuzzy metric space was introduced by O.Kramosil and J. Michalek [25] in 1975. Helpen [19] in 1981 first proved a fixed point theorem for fuzzy mappings. Also M.Grabiec [17] in 1988 proved the contraction principle in the setting of the fuzzy metric spaces. Moreover, A. George and P. Veeramani [16] in 1994 modified the notion of fuzzy metric spaces with the help of t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metric space. We know that that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. In the present paper we are proving a common fixed point theorem for fuzzy3-metric spaces for weakly commuting mappings which are compatible also.
2. SOME FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACE

Definition (2 A): A binary operation \( * \) : \([0, 1] \times [0, 1] \times [0, 1] \to [0, 1]\) is called a continuous t-norm if \((0,1,1)\) is an abelian topological monoids with unit 1 such that \( a_1 \leq a_2 \land b_1 \leq b_2 \land c_1 \leq c_2 \land d_1 \leq d_2 \) for all \( a_1, a_2, b_1, b_2, c_1, c_2 \) and \( d_1, d_2 \) are in \([0,1]\).

Definition (2 B): The 3-tuple \((X, M, \ast)\) is called a fuzzy 3-metric space if \(X\) is an arbitrary set, \(\ast\) is continuous t-norm and \(M\) is fuzzy set in \(\mathbb{X}^4 \times [0, \infty)\) satisfying the followings

\[
(M^{F \rightarrow 1}) : M(x, y, z, w, 0) = 0
\]

\[
(M^{F \rightarrow 2}): M(x, y, z, w, t) = 1, \forall t > 0
\]

\[
(M^{F \rightarrow 3}): M(x, y, z, w, t) = M(x, t, y, w) = M(z, w, x, y) = \ast
\]

\[
(M^{F \rightarrow 4}): M(x, y, z, w, t_1 + t_2 + t_3) \geq M(x, y, z, t_1) * M(y, z, t_2) * M(z, w, t_3)
\]

\[
(M^{F \rightarrow 5}): M(x, y, z, w) : [0, 1] \to 0
\]

Definition (2 C): Let \((X, M, \ast)\) be a fuzzy 3-metric space. A sequence \(\{x_n\}\) in fuzzy 3-metric space \(X\) is said to be convergent to a point \(x \in X\),

\[
\lim_{n \to \infty} M(x_n, x, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t > 0
\]

A sequence \(\{x_n\}\) in fuzzy 3-metric space \(X\) is called a Cauchy sequence, if

\[
\lim_{n \to \infty} M(x_n, x_m, a, b, t) = 1, \text{ for all } a, b \in X \text{ and } t > 0
\]

A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2 D): A function \(M\) is continuous in fuzzy 3-metric space, iff whenever for all \(a \in X\) and \(t > 0\),

\[
x_n \to x, y_n \to y, \text{ then } \lim_{n \to \infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t), \forall a, b \in X \text{ and } t > 0
\]

1. DEFINITION (2 E): TWO MAPPINGS A AND S ON FUZZY 3-METRIC SPACE X ARE WEAKLY COMMUTING IFF

\[
M(ASu, SAv, a, b, t) \geq M(Au, Su, a, b, t), \forall u, a, b \in X \text{ and } t > 0
\]

3. MAIN RESULT

Theorem 3.1 Let \((X, M, \ast)\) be a complete fuzzy 3-metric space and let \(F\) and \(T\) be continuous mappings of \(X\) in \(X\). Let \(A\) be a self mapping of \(X\) satisfying \(\{A, F\}\) and \(\{A, T\}\) are weakly commuting and

(3.1a) \(A(X) \subseteq F(X) \cap T(X)\)

(3.1b)

\[
M(Ax, Ay, a, b, t) \geq r
\]

\[
[\min\{M(Fx, Ty, a, b, t) \}, M(Fx, Ax, a, b, t), M(Fy, Ay, a, b, t)\},
\]

\[
[\min\{M(Tx, Ay, a, b, t), M(Ax, Ty, a, b, t), M(Fy, Ay, a, b, t)\}]
\]

For all \(x, y \in X\), where \(r : [0, 1] \to [0, 1]\) is a continuous function such that \(r(t) \geq t\) for each \(0 \leq t \leq 1\) and

\[
r(t) = 1 \text{ for } t = 1. \text{ Then sequence } \{x_n\}\text{ and } \{y_n\}\text{ in } X\text{ are such that}
\]

\[
x_n \to x, y_n \to y \Rightarrow M(x_n, y_n, a, b, t) \to M(x, y, a, b, t), \forall a, b \in X \text{ and } t > 0
\]

where \(t > 0\)

(3.1c)

Then \(F, T\) and \(A\) have a unique common fixed point in \(X\).

Proof: We define a sequence \(\{Ax_n\}\) such that \(Fx_{2n+1} = Ax_{2n}\) and \(Tx_{2n+2} = Ax_{2n+1}, n = 1, 2, \ldots\). We shall prove that

\[
\{Ax_n\}\text{ is a Cauchy sequence for } u = 0, 1, 2, \ldots
\]

\[
G_{2n} = M(Ax_{2n}, Ax_{2n+1}, t) \leq 1, 2, 3, \ldots
\]

\[
G_{2n} = M(Ax_{2n+1}, Ax_{2n+2}, t)
\]

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\[ M(Ax_n, Ax_{n+1}, a, b, t) \geq M(A, Ax_{n+1}, a, b, t) \]

Thus \(\{Ax_n\}\) is a Cauchy sequence and by completeness of \(X\), \(\{Ax_n\}\) converges to \(u \in X\). So subsequence \(\{T^n x_{n+1}\}\) and \(\{T^n x_{n+1}\}\) are also converges to same point \(u\).

Since \(A\) is R weakly commuting with \(F\), so

\[ M(AFx_{n+1}, FAx_{n+1}, a, b, t) \geq M(AFx_{n+1}, FAx_{n+1}, a, b, t) \]

On taking limit \(n \to \infty\), \(AFx_{n+1} = FAx_{n+1} = Fu\). Now we will prove \(Fu = u\), First suppose that \(Fu \neq u\) then exist \(t > 0\) such that \(M(Fu, u, a, t) < 1\)

Now

\[ M(Fu, u, a, t) \geq r \min \left\{ M(Fu, u, a, b, t), M(Fu, u, a, b, t), M(Fu, u, a, b, t) \right\} \]

which is a contradiction.

Thus \(u\) is a fixed point of \(F\). Similarly we can show that \(u\) is also fixed point of \(A\). Now we claim that \(u\) is a fixed point of \(T\). Suppose it is not so then for any \(t > 0\), \(M(u, Tu, a, t) < 1\)

\[ M(Au, Tx_{2n}, a, b, t) \geq r \min \left\{ M(Fu, Tu, a, b, t), M(Fu, Au, a, b, t), M(Fu, ATx_{2n}, a, b, t), M(Fu, ATx_{2n}, a, b, t) \right\} \]

which is a contradiction so \(M(Tu, u, a, b, t) = 1\).

Hence \(u\) is also a fixed point of \(T\). That is \(u\) is a common fixed point of \(T, F\) and \(A\).

Uniqueness: Suppose there is another fixed point \(v \neq u\), then

\[ M(Ax, Ay, a, b, t) \geq r \min \left\{ M(Fx, Ty, a, b, t), M(Fx, Ax, a, b, t), M(Fx, Ay, a, b, t) \right\} \]

\[ M(Au, Av, a, b, t) \geq r \min \left\{ M(Fu, Tu, a, b, t), M(Fu, Au, a, b, t), M(Fu, Av, a, b, t) \right\} \]
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\[
M(u,v,a,b,t) \geq r \left[ \min \{ M(u,v,a,b,t), M(u,u,a,b,t), M(v,v,a,b,t) \} \right] \\
M(u,v,a,b,t) \geq r \left[ \min \{ M(u,v,a,b,t), M(u,v,a,b,t) \} \right]
\]

which is a contradiction so \( u = v \).

Hence A, F and T have unique common fixed point.

REFERENCES