Some Fixed Point Theorems for Weakly Commuting Mappings Related to Fuzzy-3 Metric Spaces

Mahesh Kumar Verma, Vikas Kumar Mishra, Manoj Kumar Jha

Abstract— **In the present paper some generalization on fixed point and common fixed point theorems in complete Fuzzy 3-metric spaces are established**

Index Terms— **Fuzzy metric spaces, fuzzy 3- metric spaces, fixed point, Common fixed point.**

I. INTRODUCTION

Highlight a section that you want to designate with a certain style, The study of common fixed points of mappings in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. In1965, the concept of fuzzy sets was introduced by Zadeh [36]. With the concept of fuzzy sets, the fuzzy metric space was introduced by O.Kramosil and J. Michalek [25] in 1975. Helpern [19] in 1981first proved a fixed point theorem for fuzzy mappings. Also M.Grabiec [17] in 1988 proved the contraction principle in the setting of the fuzzy metric spaces. Moreover, A. George and P. Veeramani [16] in1994 modified the notion of fuzzy metric spaces with the help of t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metic space. We know that that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. In the present paper we are proving a common fixed point theorem for **fuzzy3-metric** spaces for weakly commuting mappings which are compatible also.

Manuscript received Oct 15, 2015

Mahesh kumar verma, Department of mathematics, Govt GB college Hardibazar, Korba, India
Vikas kumar Mishra, Department of mathematics, Rungta college of Engg. & Tech., Raipur, India
Manoj kumar jha, Department of mathematics, Rungta Engg. college, Raipur, India

2. SOME FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACE

Definition (2 A): A binary operation *: [0, 1] x [0,1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 * d \geq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$ for all a_1, a_2 , b_1, b_2, c_1, c_2 and d_1, d_2 are in [0,1].

Definition (2 B): The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in $X^4 \times [0,\infty)$ satisfying the followings

$(FM'' - 1)$: $M(x,y,z,w,0) = 0$

$(FM'' - 2)$: $M(x,y,z,w,t) = 1, \forall\, t > 0$

$(FM'' - 3)$: $M(x,y,z,w,t) = M(x,w,z,y,t) = M(z,w,x,y,t) = \cdots$

$(FM'' - 4)$: $M(x,y,z,w,t_1 + t_2 + t_3) \geq M(x,y,z,u,t_1) * M(x,y,u,w,t_2) * M(x,u,z,w,t_3) * M(x,y,z,w,t_4)$

$(FM'' - 5)$: $M(x,y,z,w)$: $[0,1) \to [0,1]$ $is\ left\ continuous, \forall\ x,y,z,u \in X, t_1,t_2,t_3,t_4 > 0$

Definition (2 C): Let (X, M, *) be a fuzzy 3-matric space. A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$,

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1, for\ all\ a, b \in X\ and\ t > 0$$

A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1, for\ all\ a, b \in X\ and\ t, p > 0$$

A fuzzy 3-matric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2 D): A function M is continuous in fuzzy 3-metric space, iff whenever for all $a \varepsilon X$ and $t > 0$.

$$x_n \to x, y_n \to y, then \lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x,y,a,t), \forall\ a, b \in X\ and\ t > 0$$

I. DEFINITION (2 E): TWO MAPPINGS A AND S ON FUZZY 3-METRIC SPACE X ARE WEAKLY COMMUTING IFF

$$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t), \forall u, a, b \in X\ and\ t > 0$$

3. MAIN RESULT

Theorem 3.1 Let $(X, M, *)$ be a complete fuzzy 3-metric space and let F and T be continuous mappings of X in X. Let A be a self mapping of X satisfying {A,F} and {A,T} are weakly commuting and

(3.1a) $A(X) \subseteq F(X) \cap T(X)$

(3.1b)

$$M(Ax, Ay, a, b, t) \geq r\begin{bmatrix} Min\{M(Fx,Ty,a,b,t)\}, M(Fx,Ax,a,b,t), M(Fx,Ay,a,b,t), \\ M(Ty,Ay,a,b,t), M(Ax,Ty,a,b,t), M(Fy,Ay,a,b,t)\} \end{bmatrix}$$

For all $x, y \subset X$, where r: $[0,1] \to [0,1]$ is a continuous function such that $r(t) > t$ for each $0 \leq t \leq 1$ and $r(t) = 1$ for $t = 1; a, b \in X$. Then sequence $\{x_n\}$ and $\{y_n\}$ in X are such that

$$x_n \to x, y_n \to y \Longrightarrow M(x_n, y_n, a, b, t) \longrightarrow M(x,y,a,b,t)$$

where $t > 0$ $\qquad (3.1c)$

Then F,T and A have a unique common fixed point in X.

Proof: We define a sequence $\{x_n\}$ such that $Fx_{2n+1} = Ax_{2n}$ and $Tx_{2n+2} = Ax_{2n+1}, n = 1,2,\ldots$ We shall prove that $\{Ax_n\}$ is a Cauchy sequence for $n = 0,1,2,\ldots$

$$G_n = M(Ax_n, Ax_{n+1}, t) < 1; n = 0,1,2,3,\ldots\ldots$$

$$G_{2n} = M(Ax_{2n+1}, Ax_{2n}, t)$$

$$\geq r\begin{bmatrix} Min\{M(Fx_{2n+1},Tx_{2n},a,b,t)\},M(Fx_{2n+1},Ax_{2n+1},a,b,t),M(Fx_{2n+1},Ax_{2n},a,b,t), \\ M(Tx_{2n},Ax_{2n},a,b,t),M(Ax_{2n+1},Tx_{2n},a,b,t),M(Fx_{2n},Ax_{2n},a,b,t)\} \end{bmatrix}$$
$$= r\begin{bmatrix} Min\{M(Ax_{2n},Ax_{2n-1},a,b,t)\},M(Ax_{2n},Ax_{2n+1},a,b,t),M(Ax_{2n},Ax_{2n},a,b,t), \\ M(Ax_{2n-1},Ax_{2n},a,b,t),M(Ax_{2n+1},Ax_{2n-1},a,b,t),M(Ax_{2n-1},Ax_{2n},a,b,t)\} \end{bmatrix}$$
$$\geq r\begin{bmatrix} Min\{M(Ax_{2n},Ax_{2n-1},a,b,t)\},M(Ax_{2n},Ax_{2n+1},a,b,t),M(Ax_{2n},Ax_{2n},a,b,t), \\ M(Ax_{2n-1},Ax_{2n},a,b,t),M(Ax_{2n+1},Ax_{2n},a,b,t),M(Ax_{2n-1},Ax_{2n},a,b,t),M(Ax_{2n-1},Ax_{2n},a,b,t)\} \end{bmatrix}$$
$$= r[Min\{G_{2n-1},G_{2n},1,G_{2n-1},G_{2n},G_{2n-1},G_{2n-1}\}]$$

If $G_{2n-1} > G_{2n}$, then $G_{2n} > r[G_{2n-1}] > G_{2n-1}$

A contradiction, therefore $G_{2n-1} < G_{2n}$

Therefore $G_{2n} \geq r[G_{2n-1}] \geq G_{2n-1}$

Thus $\{G_{2n} : n \geq 0\}$ is increasing sequence of positive real numbers in [0,1] and therefore tends to a finite limit $L \leq 1$. It is clear that L=1 because if $L < 1$ then on taking limit $n \to \infty$ we get $L \geq r(L) > L$, a contradiction. Hence L=1.

Now for any integer m,

$$M(Ax_n,Ax_{n+m},a,b,t) \geq M\left(Ax_n,Ax_{n+1},a,b,\frac{t}{m}\right) * \dots\dots\dots * M\left(Ax_{n+m-1},Ax_{n+m},a,b,\frac{t}{m}\right)$$
$$\geq M\left(Ax_n,Ax_{n+1},a,b,\frac{t}{m}\right) * \dots\dots\dots * M\left(Ax_n,Ax_{n+1},a,b,\frac{t}{m}\right)$$

$$limit\ n \to \infty M(Ax_n,Ax_{n+m},a,b,t) \geq 1 * 1 * 1 * \dots\dots\dots * 1 = 1$$

Thus $\{Ax_n\}$ is a Cauchy sequence and by completeness of X, $\{Ax_n\}$ converges to $u \in X.$ So subsequence $\{Fx_{2n+1}\}$ and $\{Tx_{2n}\}$ of $\{Ax_n\}$ are also converges to same point u.

Since A is R weakly commuting with F, so

$$M(AFx_{2n+1},FAx_{2n+1},a,b,t) \geq M\left(AFx_{2n+1},FAx_{2n+1},a,b,\frac{t}{R}\right)$$

On taking limit $n \to \infty$, $AFx_{2n+1} = FAx_{2n+1} = Fu.$ Now we will prove $Fu = u.$ First suppose that $Fu \neq u$ then there exist $t > 0$ such that $M(Fu,u,t) < 1$

Now

$$M(AFx_{2n+1},Fx_{2n},a,b,t) \geq$$
$$r\left[Min\begin{Bmatrix} M(F^2x_{2n+1},Tx_{2n},a,b,t),M(F^2x_{2n+1},AFx_{2n+1},a,b,t),M(F^2x_{2n+1},Ax_{2n},a,b,t), \\ M(Tx_{2n},Ax_{2n},a,b,t),M(AFx_{2n+1},Tx_{2n},a,b,t),M(Fx_{2n},Ax_{2n},a,b,t) \end{Bmatrix}\right]$$
$$M(Fu,u,a,t) \geq r\left[Min\begin{Bmatrix} M(Fu,u,a,b,t),M(Fu,Fu,a,b,t),M(Fu,u,a,b,t), \\ M(u,u,a,b,t),M(Fu,u,a,b,t),M(Fu,u,a,b,t) \end{Bmatrix}\right]$$
$$M(Fu,u,a,b,t) \geq r[M(Fu,u,a,b,t)] >$$
$$M(Fu,u,a,b,t)$$

which is a contradiction.

Thus u is a fixed point of F. Similarly we can show that u is also fixed point of A. Now we claim that u is a fixed point of T. Suppose it is not so then for any t>0, $M(u,Tu,a,t) < 1$

$$M(Au,Tx_{2n},a,b,t)$$
$$\geq r\left[Min\begin{Bmatrix} M(Fu,T^2x_{2n},a,b,t),M(Fu,Au,a,b,t),M(Fu,ATx_{2n},a,b,t), \\ M(T^2x_{2n},ATx_{2n},a,b,t),M(Au,T^2x_{2n},a,b,t),M(FTx_{2n},ATx_{2n},a,b,t) \end{Bmatrix}\right]$$
$$M(u,Tu,a,b,t) \geq r\left[Min\begin{Bmatrix} M(u,Tu,a,b,t),M(u,u,a,b,t),M(u,Tu,a,b,t), \\ M(Tu,Tu,a,b,t),M(u,Tu,a,b,t),M(Tu,Tu,a,b,t) \end{Bmatrix}\right]$$
$$M(u,Tu,a,b,t) \geq r[Min\{M(u,Tu,a,b,t)\}]$$

Which is a contradiction so $M(Tu,u,a,b,t) = 1$

Hence u is also a fixed point of T. That is u is a common fixed point of T, F and A.

Uniqueness: Suppose there is another fixed point $v \neq u,$ then

$$M(Ax,Ay,a,b,t) \geq r\left[Min\begin{Bmatrix} M(Fx,Ty,a,b,t),M(Fx,Ax,a,b,t),M(Fx,Ay,a,b,t), \\ M(Ty,Ay,a,b,t),M(Ax,Ty,a,b,t),M(Fy,Ay,a,b,t) \end{Bmatrix}\right]$$
$$M(Au,Av,a,b,t) \geq r\left[Min\begin{Bmatrix} M(Fu,Tv,a,b,t),M(Fu,Au,a,b,t),M(Fu,Av,a,b,t), \\ M(Tv,Av,a,b,t),M(Au,Tv,a,b,t),M(Fv,Av,a,b,t) \end{Bmatrix}\right]$$

$$M(u,v,a,b,t) \geq r\left[Min\begin{Bmatrix} M(u,v,a,b,t), M(u,u,a,b,t), M(u,v,a,b,t), \\ M(v,v,a,b,t), M(u,v,a,b,t), M(v,v,a,b,t) \end{Bmatrix}\right]$$

$M(u,v,a,b,t) \geq r[Min\{M(v,v,a,b,t)\}]$, which is a contradiction so u=v.

Hence A, F and T have unique common fixed point.

References

[1] Badard, R. "Fixed point theorems for fuzzy numbers" Fuzzy sets and systems 13 (1984) 291-302.

[2] Bose, B.K. and Sahani, D. "Fuzzy mappings and fixed point theorems" Fuzzy sets and Systems 1 (1984) 53-58.

[3] Butnariu, D. "Fixed point for fuzzy mappings" Fuzzy sets and Systems 7 (1982) 191-207.

[4] Chang, S.S. "Fixed point theorems for fuzzy mappings" Fuzzy Sets and Systems 17(1985) 181-187.

[5] Change, S.S., Cho, Y.J., Lee, B.S. and Lee, G.M. "Fixed degree and fixed point theorems for fuzzy mappings" Fuzzy Sets and Systems 87 (1997) 325-334.

[6] Change, S.S., Cho, Y.J. , Lee B.S., June, J.S. and Kang, S.M. "Coincidence point and minimization theorems in fuzzy metric spaces" Fuzzy Sets and Systems, 88 (1) (1997) 119-128.

[7] Cho, Y.J. "Fixed points and fuzzy metric spaces" J. Fuzzy Math.5 (1997) no.4, 949-962.

[8] Deng, Z. "Fuzzy pseudo-metric space" J. Math. Anal. Appl.86 (1982) 74-95.

[9] Ekland, I. and Gahler, S. "Basic notions for fuzzy topology" Fuzzy Sets and System 26 (1988) 336-356.

[10] Erceg, M.A. "Metric space in fuzzy set theory" J. Math. Anal. Appl.69 (1979) 205-230.

[11] Fang, J.X. "On fixed point theorems in fuzzy metric spaces" Fuzzy Sets and Systems 46 (1979) 107-113.

[12] Fisher, B. "Mappings with a common fixed point" Math. Seminar notes Kobe university 7 (1979) 81-84.

[13] Gahler, S. Linear 2-normierte Raume" Math. Nachr. 28 (1964) 1-43.

[14] Gahler, S. Uber 2-Banach Raume, Math. Nachr. 42 (1969) 335-347

[15] Gahler, S. 2- metrische Raume and ihre topologische structure, Math. Nachr.26 (1983) 115-148.

[16] George, A. and Veramani, P. "On some results in fuzzy metric spaces" Fuzzy Sets and Systems 64 (1994) 395-399.

[17] Grabiec, M. "Fixed points in fuzzy metric space" Fuzzy Sets and Systems 27 (1988) 385-389.

[18] Hu, C. "Fuzzy topological space" J. Math. Anal. Appl. 110(1985)141-178.

[19] Heipern, S. "Fuzzy mappings and fixed point theorems" J. Math. Anal. Appl. 83 (1981) 566-569.

[20] Hadzic, O. "Fixed point theorems, for multi-valued mappings in some classes of fuzzy metric space" Fuzzy Sets and Systems 29(1989) 115-125.

[21] Jung, J.S., Cho. Y.J. and Kim J.K.: Minimization theorems for fixed point theorems in fuzzy metric spaces and applications, Fuzzy Sets and Systems 61 (1994) 199-207.

[22] Jung, J.S., Cho, Y.J., Chang, S.S. and Kang, S.M. "Coincidence theorems for set valued mappings and Ekland's variational principle in fuzzy metric spaces" Fuzzy Sets and Systems 79 (1996) 239-250.

[23] Kaleva, O. and Seikkala, S. "On Fuzzy metric spaces" Fuzzy Sets and Systems 12 (1984) 215-229.

[24] Kaleva, O. "The Completion of fuzzy metric spaces" J. Math. Anal. Appl. 109(1985) 194-198.

[25] Kramosil, I. and Michalek, J. "Fuzzy metric and statistical metric spaces" Kybernetica 11 (1975) 326-334.

[26] Lee, B.S., Cho, Y.J. and Jung, J.S. "Fixed point theorems for fuzzy mappings and applications" Comm. Korean Math. Sci. 11 (1966) 89-108.

[27] Kutukcu, S., Sharma, S. and Tokgoz, H. "A Fixed point theorem in fuzzy metric spaces" Int. Journal of Math. Analysis 1 (2007) no.18, 861-872.

[28] Mishra, S.N., Sharma, N. and Singh, S.L. "Common fixed points of maps on fuzzy metric spaces" Internet. J. Math. & Math Sci.17 (1966) 89-108.

[29] Schweitzer, B. and Sklar, A. "Statistical metric spaces" Pacific Journal Math.10 (1960) 313-334.

[30] Sharma. P.L., Sharma. B.K. and Iseki, K. "Contractive type mapping on 2- metric spaces" Math. Japonica 21 (1976) 67-70.

[31] Sharma, S. "On fuzzy Metric space" Southeast Asian Bulletin of Mathematics 26 (2002) 133-145.

[32] Som, T. "Some results on common fixed point in fuzzy Metric spaces" Soochow Journal of Mathematics 4 (2007) 553-561.

[33] Turkoglu, D and Rhodes, B.E. "A fixed fuzzy point for fuzzy mapping in complete metric spaces" Mathematical Communications10 (2005)115-121.

[34] Yadava, R.N., Rajput, S.S. Choudhary, S. and Bhardwaj, R.K "Some fixed point theorems for rational inequality in 2- metric spaces" Acta Ciencia Indica 33 (2007) 709-714.

[35] Yadava, R.N., Rajput, S.S. Choudhary, S. and Bhardwaj, R.K. "Some fixed point theorems for non contraction type mapping on 2- Banach spaces" Acta Ciencia Indica 33 (2007) 737-744.

[36] Zadeh, L.A. "Fuzzy Sets" Information and control 8 (1965) 338-353