

Cartesian Product of K- Hsu-Structure Manifold

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Abstract— Cartesian product of two manifolds has been defined and studied by Pandey [1]. In this paper we have taken Cartesian product of k-Hsu-structure manifolds, where k is some finite integer, and studied some properties of curvature and Ricci tensor of such a product manifold.

Index Terms—: k-Hsu-structure Manifolds, generalized almost contact structure, KH-structure.

I. INTRODUCTION

Let M_1, M_2, \dots, M_k be k-Hsu-structure manifolds each of class C^∞ and of dimension n_1, n_2, \dots, n_k respectively. Suppose

$(M_1)m_1, (M_2)m_2, \dots, (M_k)m_k$ be their tangent spaces at

$m_1 \in M_1, m_2 \in M_2, \dots, m_k \in M_k$, then the product space

$(M_1)m_1 \times (M_2)m_2 \times \dots \times (M_k)m_k$ contains vector fields of the form (X_1, X_2, \dots, X_k) , where

$X_1 \in (M_1)m_1, X_2 \in (M_2)m_2, \dots, X_k \in (M_k)m_k$. Vector addition and scalar multiplication on above product space are defined as follows :

$$(1.1) \quad (X_1, X_2, \dots, X_k) + (Y_1, Y_2, \dots, Y_k) = (X_1 + Y_1, X_2 + Y_2, \dots, X_k + Y_k),$$

$$(1.2) \quad \lambda(X_1, X_2, \dots, X_k) = (\lambda X_1, \lambda X_2, \dots, \lambda X_k)$$

where $X_i, Y_i \in (M_i)m_i, i = 1, 2, \dots, k$ and λ is a scalar.

Under these conditions the product space

$(M_1)m_1 \times (M_2)m_2 \times \dots \times (M_k)m_k$ forms a vector space.

Define a linear transformation F on the product space

(1.3)

$$F(X_1, X_2, \dots, X_k) = (F_1 X_1, F_2 X_2, \dots, F_k X_k)$$

where F_1, F_2, \dots, F_k are linear transformations on $(M_1)m_1, (M_2)m_2, \dots, (M_k)m_k$ respectively.

If f_1, f_2, \dots, f_k be C^∞ functions over the spaces $(M_1)m_1, (M_2)m_2, \dots, (M_k)m_k$ respectively,

we define the C^∞ function f_1, f_2, \dots, f_k on the product space as

$$(1.4) \quad (f_1, f_2, \dots, f_k) = (X_1 f_1, X_2 f_2, \dots, X_k f_k)$$

Let D_1, D_2, \dots, D_k be the connections on the manifolds M_1, M_2, \dots, M_k respectively. We define the operator D on the product space as

$$(1.5) \quad (D(X_1, X_2, \dots, X_k) (Y_1, Y_2, \dots, Y_k)) = (D_{X_1} Y_1, D_{X_2} Y_2, \dots, D_{X_k} Y_k)$$

Then D satisfies all four properties of a connection and thus it is a connection on the product manifold.

II. SOME RESULTS

Definition Let there be defined on V_n , a vector valued linear function F of class C such that

$$F^2 = a^r I_n \quad 0 \leq r \leq n$$

where r is an integer and a is real or imaginary number. Then F is called Hsu – structure and V_n is called the **Hsu – structure manifold**.

Theorem 2.1 : The product manifold

$M_1 \times M_2 \times \dots \times M_k$ admits a Hsu-structure if and only if the manifolds M_1, M_2, \dots, M_k are Hsu-structure manifolds.

Proof: Suppose M_1, M_2, \dots, M_k are Hsu-structure manifolds. Thus there exist tensor fields

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F_1, F_2, \dots, F_k each of type (1,1) on M_1, M_2, \dots, M_k respectively satisfying

$$(2.1) \quad F^2_i(X_i) = a^r X_i, \quad i = 1, 2, \dots, k$$

Where a is any complex number, not equal to zero.

In view of equation (1.3) it follows that there exists a linear transformation F on $M_1 \times M_2 \times \dots \times M_k$ satisfying

$$(2.2) \quad \begin{aligned} F^2(X_1, X_2, \dots, X_k) &= (F_1^2 X_1, F_2^2 X_2, \dots, F_k^2 X_k) \\ &= a^r (X_1, X_2, \dots, X_k) \end{aligned}$$

Thus, the product manifold admits a Hsu-structure.

Let us define a Riemannian metric g on the product manifold $M_1 \times M_2 \times \dots \times M_k$ as

$$(2.3) \quad \begin{aligned} a^r g((X_1, X_2, \dots, X_k), (Y_1, Y_2, \dots, Y_k)) &= a^r g_1(X_1, Y_1) + a^r g_2(X_2, Y_2) \\ &+ \dots + a^r g_k(X_k, Y_k), \end{aligned}$$

where g_1, g_2, \dots, g_k are the Riemannian metrics over the manifolds M_1, M_2, \dots, M_k respectively.

If $\xi_1, \xi_2, \dots, \xi_k$ be vector fields and $\eta_1, \eta_2, \dots, \eta_k$ be 1-forms on the Hsu-structure manifolds

M_1, M_2, \dots, M_k respectively, then a vector field ξ and a 1-form η on the product manifold

M_1, M_2, \dots, M_k is defined.

We now prove the following results.

Theorem 2.2 : The product manifold $M_1 \times M_2 \times \dots \times M_k$ admits a generalized almost contact structure if and only if the manifolds M_1, M_2, \dots, M_k possess the same structure.

Proof : Let M_1, M_2, \dots, M_k are generalized almost contact manifolds. Thus there exists tensor fields F_i of type (1,1) vector fields ξ_i and 1-form η_i , $i = 1, 2, \dots, k$ satisfying

$$(2.4) \quad F_i^2(X_i) = a^r X_i + \eta_i(X_i) \xi_i,$$

For product manifold $M_1 \times M_2 \times \dots \times M_k$.

$$F^2(X_1, X_2, \dots, X_k) = (F_1^2 X_1, F_2^2 X_2, \dots, F_k^2 X_k)$$

By the help of equation (2.4), takes the form

$$F^2(X_1, X_2, \dots, X_k) = a^r (X_1, X_2, \dots, X_k) + (\eta_1(X_1)\xi_1, \eta_2(X_2)\xi_2, \dots, \eta_k(X_k)\xi_k),$$

or

$$(2.5) \quad F^2(X) = a^r X + \eta(X)\xi.$$

Hence the product manifold admits a generalized almost contact metric structure. [2].

Theorem 2.3 : The product manifold $M_1 \times M_2 \times \dots \times M_k$ admits a KH-structure if and only if the manifolds M_1, M_2, \dots, M_k are KH-structure manifolds.

Proof: Suppose M_1, M_2, \dots, M_k are KH-structure manifolds. Thus

$$(2.6) \quad \begin{aligned} (D_{1_{X_1}} F_1)(Y_1) &= (D_{2_{X_2}} F_2)(Y_2) \\ &= \dots \\ &= (D_{k_{X_k}} F_k)(Y_k) \\ &= 0 \end{aligned}$$

As D is a connection on the product manifold, we have

$$(2.7) \quad (D(X_1, X_2, \dots, X_k)F)(Y_1, Y_2, \dots, Y_k) = D(X_1, X_2, \dots, X_k)\{F(Y_1, Y_2, \dots, Y_k)\} - F\{D(X_1, X_2, \dots, X_k)(Y_1, Y_2, \dots, Y_k)\}$$

In view of equation (1.3) and equation (1.5), this takes the form

$$\begin{aligned} (D(X_1, X_2, \dots, X_k)F)(Y_1, Y_2, \dots, Y_k) &= D(X_1, X_2, \dots, X_k)(F_1 Y_1, F_2 Y_2, \dots, F_k Y_k) \\ &\quad - F(D_{1_{X_1}} Y_1, D_{2_{X_2}} Y_2, \dots, D_{k_{X_k}} Y_k) \\ &= -(D_{1_{X_1}} F_1 Y_1, D_{2_{X_2}} F_2 Y_2, \dots, D_{k_{X_k}} F_k Y_k) \\ &\quad - (F_1 D_{1_{X_1}} Y_1, F_2 D_{2_{X_2}} Y_2, \dots, F_k D_{k_{X_k}} Y_k) \\ &= ((D_{1_{X_1}} F_1)(Y_1), (D_{2_{X_2}} F_2)(Y_2), \dots, (D_{k_{X_k}} F_k)(Y_k)) \\ &= 0. \end{aligned}$$

Thus, the product manifold is KH-structure manifold.

Theorem 2.4: The product manifold $M_1 \times M_2 \times \dots \times M_k$ of Hsu-structure manifolds M_1, M_2, \dots, M_k is almost Tachibana if and only if the manifolds M_1, M_2, \dots, M_k are separately Tachibana manifolds.

Proof: Let a Hsu-structure manifolds M_1, M_2, \dots, M_k are almost Tachibana manifolds. Then

$$(2.8) \quad (D_{i_{X_i}} F_i)(Y_i) + (D_{i_{Y_i}} F_i)(Y_i) = 0, \quad i = 1, 2, \dots, k.$$

3. Curvature and Ricci Tensor

Let $X = (X_1, X_2, \dots, X_k)$ and $Y = (Y_1, Y_2, \dots, Y_k)$ be C^∞ vector fields on the product manifold

$M_1 \times M_2 \times \dots \times M_k$ and $F = (f_1, f_2, \dots, f_k)$ be a C^∞ function. Then

$$(3.1) \quad \begin{aligned} &[(X_1, X_2, \dots, X_k), (Y_1, Y_2, \dots, Y_k)](f_1, f_2, \dots, f_k) \\ &= (X_1, X_2, \dots, X_k)\{(Y_1, Y_2, \dots, Y_k)(f_1, f_2, \dots, f_k)\} - (Y_1, Y_2, \dots, Y_k) \\ &\quad = ([X_1, Y_1]f_1, [X_2, Y_2]f_2, \dots, [X_k, Y_k]f_k). \end{aligned}$$

Suppose $K_i(X_i, Y_i, Z_i)$, $i = 1, 2, \dots, k$ be the curvature tensors of the Hsu-structure manifolds

M_1, M_2, \dots, M_k respectively. If $K(X, Y, Z)$ be the curvature tensor of the product manifold

$M_1 \times M_2 \times \dots \times M_k$. Then we have

$$(3.2) \quad K(X, Y, Z) = [K_1(X_1, Y_1, Z_1), K_2(X_2, Y_2, Z_2), \dots, K_k(X_k, Y_k, Z_k)].$$

If $W = (W_1, W_2, \dots, W_k)$ be a vector field on the product manifold, then

$$(3.3) \quad K'(X, Y, Z, W) = g(K(X, Y, Z, W)),$$

$$(3.4) \quad \begin{aligned} K'(X, Y, Z, W) &= K'_1(X_1, Y_1, Z_1, W_1) + K'_2(X_2, Y_2, Z_2, W_2) + \dots \\ &\quad + K'_k(X_k, Y_k, Z_k, W_k) \end{aligned}$$

Thus, we have

Theorem 3.1 : The product of manifold $M_1 \times M_2 \times \dots \times M_k$ is of constant curvature if and only if Hsu-structure manifolds M_1, M_2, \dots, M_k are separately of constant curvature.

Theorem 3.2 : The Ricci tensor of the product manifold $M_1 \times M_2 \times \dots \times M_k$ is the sum of the Ricci tensor of the Hsu-structure manifolds M_1, M_2, \dots, M_k .

Theorem 3.3 : The product of manifold $M_1 \times M_2 \times \dots \times M_k$ is an Einstein space if and only if the Hsu-structure manifolds M_1, M_2, \dots, M_k are separately Einstein space.

Proof : Let the product manifold $M_1 \times M_2 \times \dots \times M_k$ be an Einstein space. Thus

$$(3.5) \quad Ric(X, Y) = Cg(X, Y),$$

Where $C = \frac{K}{n}$, K being the scalar curvature and n being the dimension of the product manifold. Then

$$Ric(X_i, Y_i) = Cg_i(X_i, Y_i), \quad i = 1, 2, \dots, k.$$

Therefore the manifolds M_1, M_2, \dots, M_k are also Einstein spaces.

References

- [1] H. B. Pandey, Cartesian product of two manifolds, Indian Journ. Pure Appl. Math, 12(1) (1981) 55-60.
- [2] J. Pant, Hypersurface immersed in a GF-structure manifold, Demonstratio Mathematica, 19(3) (1986) 693-697.