International Journal of Engineering Research And Management (IJERM)
ISSN : 2349- 2058, Volume-03, Issue-03, March 2016

Complex of Lascoux in Partition (6,5,3)

Haytham Razooki Hassan, Noor TahaAbd

***Abstract*—In this paper ,the complex of Lascoux in the case of partition (6,5,3) has been studied by using diagram divided power of the place polarization$\partial_{ij}^{(k)}$,Capelli identities and the idea of mapping Cone.**

***Index Terms*—Divided power algebra, Resolution of Weyl module ,place polarization, Mapping Cone.**

I. INTRODUCTION

Let R be a commutative ring with 1,F be free R-module and D_nF be the divided power of degree n[1]. consider the map $D_{p+k}F\otimes D_{q-k}F \to D_pF\otimes D_qF$, this map is a place polarization from place one to place two where place one and two.

are denoted by $D_{p+k}F$ *and* $D_{q-k}F$ respectively ,and the map $\partial_{32}^{(k)}: D_pF\otimes D_{q+k}F\otimes D_{r-k}F \to D_pF\otimes D_qF\otimes D_rF$is the place polarization from place twoTo place three [2]-[4]. In this point we can also ask for the identities in case such that $\partial_{21}\partial_{32}$ Where three places have been looked , so we get to use the following equation

$$\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{21}^{(1)} \circ \partial_{32}^{(1)} = \partial_{31}^{(1)} \qquad (1.1)$$

This is a typical **Capelli identity** [5][6][2] More than $\partial_{32}^{(k)} \circ \partial_{21}^{(l)} = \sum_{\alpha\geq 0} \partial_{21}^{(l-\alpha)} \circ \partial_{32}^{(k-\alpha)} \circ \partial_{31}^{(\alpha)}$ *where* $\partial_{ij}^{(0)} = I$ (1.2)

In general the divided of a place polarizations satisfy the following identities in case of $k \neq i$

$$\partial_{ij}^{(r)} \circ \partial_{jk}^{(s)} = \sum_{\alpha\geq 0} \partial_{jk}^{(s-\alpha)} \circ \partial_{ij}^{(r-\alpha)} \circ \partial_{ik}^{(\alpha)} \qquad (1.3)$$

$$\partial_{jk}^{(s)} \circ \partial_{ij}^{(r)} = \sum_{\alpha\geq 0} (-1)^{\alpha}\, \partial_{ij}^{(r-\alpha)} \circ \partial_{jk}^{(s-\alpha)} \circ \partial_{ik}^{(\alpha)} \qquad (1.4)$$

$$\partial_{21}^{(l)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{21}^{(k)} \qquad (1.5)$$

$$\partial_{32}^{(k)} \circ \partial_{31}^{(1)} = \partial_{31}^{(1)} \circ \partial_{32}^{(k)} \qquad (1.6)$$

Buchsbaum in 2004 modified the boundary method [5]. he and author [6]. studied the complex Of Lascoux (characteristic zero)in the partition (2,2,2),(3,3,3)'respectively, also the author in[7]. studied a complex of Lascoux (characteristic zero)in the partition(4,4,3)using this modified method and a technique of Letter place methods[7]. and Mapping cone[8]. obtain characteristic zero.

II. THE TERMS OF LASCOUX COMPLEX IN THE CASE OF PARTITION(6,5,3)

A. The terms

The terms of the Lascoux complex are obtained from the determinantal expansion of the Jacobi-Trudi matrix of the partition .The position of the terms of the complex are determined by The length of the permutation to which they correspond [5][6]. Now in the case of the partition $\lambda = (6,5,3)$ we have the following matrix:

$$\begin{bmatrix} D_6F & D_4F & D_1F \\ D_7F & D_5F & D_2F \\ D_8F & D_6F & D_3F \end{bmatrix}$$

Then the Lascoux complex has the correspondence between It's terms as follows:

$D_6F\otimes D_5F\otimes D_3F \leftrightarrow$ identity
$D_4F\otimes D_7F\otimes D_3F \leftrightarrow (12)$
$D_6F\otimes D_2F\otimes D_6F \leftrightarrow (23)$
$D_4F\otimes D_2F\otimes D_8F \leftrightarrow (123)$
$D_1F\otimes D_5F\otimes D_8F \leftrightarrow (13)$
$D_1F\otimes D_6F\otimes D_7F \leftrightarrow (132)$

So ,the complex of Lascoux in the case of the partition λ=(6,5,3) has the form:-

$$D_8F\otimes D_5F\otimes D_1F \to \begin{array}{c} D_8F\otimes D_4F\otimes D_2F \\ \oplus \\ D_7F\otimes D_6F\otimes D_1F \end{array} \to \begin{array}{c} D_7F\otimes D_4F\otimes D_3F \\ \oplus \\ D_6F\otimes D_6F\otimes D_2F \end{array} \to D_6F\otimes D_5F\otimes D_3F$$

B. The Complex Of Lascoux As A diagram

Consider the following diagram:

$$\begin{tikzcd}
D_8F\otimes D_5F\otimes D_1F \arrow[r, "n_1"] \arrow[d, "k_1"'] \arrow[dr, phantom, "S"] & D_8F\otimes D_4F\otimes D_2F \arrow[r, "n_2"] \arrow[d, "k_2"'] \arrow[dr, phantom, "H"] & D_7F\otimes D_4F\otimes D_3F \arrow[d, "k_3"'] \\
D_7F\otimes D_6F\otimes D_1F \arrow[r, "m_1"] & D_6F\otimes D_6F\otimes D_2F \arrow[r, "m_2"] & D_6F\otimes D_5F\otimes D_3F
\end{tikzcd}$$

So, if we define

$n_1: D_8F\otimes D_5F\otimes D_1F \to D_8F\otimes D_4F\otimes D_2F$

as $n_1(v) = \partial_{32}^{(1)}(v)$ *Where* $v \in D_8F\otimes D_5F\otimes D_1F$

$k_1: D_8F \otimes D_5F \otimes D_1F \to D_7F \otimes D_6F \otimes D_1F$

as $k_1(v) = \partial_{21}^{(1)}(v)$; *Where* $v \in D_8F \otimes D_5F \otimes D_1F$

and

$k_2: D_8F \otimes D_4F \otimes D_2F \to D_6F \otimes D_6F \otimes D_2F$

as $k_2(v) = \partial_{21}^{(2)}(v)$; *where* $v \in D_8F \otimes D_4F \otimes D_2F$

Now ,we have to define the map

$m_1: D_7F \otimes D_6F \otimes D_1F \to D_6F \otimes D_6F \otimes D_2F$

Which makes the diagram S commutative i.e. $m_1 \circ k_1 = k_2 \circ n_1$

Which implies that $m_1 \circ \partial_{21}^{(1)} = \partial_{21}^{(2)} \circ \partial_{32}^{(1)}$

Manuscript received Oct 15, 2015

Dr.HaythamRazooki Hassan, Department of Mathematics, Roma University / College of science / Baghdad, Iraq.

Noor TahaAbd, Department of Mathematics, Al -Mustansiriya University/ Collegeof Science/ Baghdad, Iraq,

Now if we use Capelli identities (1.2) we get: $\partial_{21}^{(2)} \circ \partial_{32}^{(1)} = \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{31}^{(1)} \circ \partial_{21}^{(1)}$

$= (\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}) \circ \partial_{21}^{(1)}$

Thus, $m_1 = \frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}$

On the other hand, if we define

$m_2 : D_6F \otimes D_6F \otimes D_2F \rightarrow D_6F \otimes D_5F \otimes D_3F$

as $m_2(v) = \partial_{32}^{(1)}(v)$; $where\ v \in D_6F \otimes D_6F \otimes D_2F$

and $k_3 : D_7F \otimes D_4F \otimes D_3F \rightarrow D_6F \otimes D_5F \otimes D_3F$

as $k_3(v) = \partial_{21}^{(1)}(v)$; $wherev \in D_7F \otimes D_4F \otimes D_3F$

Now we need to define $n_2$to make the diagram H commute $n_2 : D_8F \otimes D_4F \otimes D_2F \rightarrow D_7F \otimes D_4F \otimes D_3F$ Such that

$k_3 \circ n_2 = m_2 \circ k_2$ i.e. $\partial_{21}^{(1)} \circ n_2 = \partial_{32}^{(1)} \circ \partial_{21}^{(2)}$

Again by using Capelli identities we have

$\partial_{32}^{(1)} \circ \partial_{21}^{(2)} = \partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)}$

$= \partial_{21}^{(1)} \circ (\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})$

Then $n_2 = \frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}$

C. Diagram

Now consider the following diagram:

$$\begin{tikzcd}
D_8F\otimes D_5F\otimes D_1F \arrow[r, "n_1"] \arrow[d, "k_1"', "E"] & D_8F\otimes D_4F\otimes D_2F \arrow[r, "n_2"] & D_7F\otimes D_4F\otimes D_3F \arrow[d, "k_3", "F"'] \\
D_7F\otimes D_6F\otimes D_1F \arrow[r, "m_1"] \arrow[urr, "p"] & D_6F\otimes D_6F\otimes D_2F \arrow[r, "m_2"] & D_6F\otimes D_5F\otimes D_3F
\end{tikzcd}$$

Define p: $D_7F \otimes D_6F \otimes D_1 \rightarrow D_7F \otimes D_4F \otimes D_3F$

By $\mathrm{p}(v) = \partial_{32}^{(2)}(v)$; $where\ v \in D_7F \otimes D_6F \otimes D_1F$

III. MATH

Proposition 1.The diagram E is commutative .

Proof:

To prove E is commutative, we need to prove $n_2 \circ n_1 = \mathrm{P} \circ k_1$

$n_2 \circ n_1 = (\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}) \circ \partial_{32}^{(1)}$

$= \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{31}^{(1)} \circ \partial_{32}^{(1)}$ from(1.2) we have

$= \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} + \partial_{31}^{(1)} \circ \partial_{32}^{(1)}$

$= \partial_{32}^{(2)} \circ \partial_{21}^{(1)}$

$= P \circ k_1$ □

Proposition 2. The diagram F is commutative .

Proof :

$m_2 \circ m_1 = \partial_{32}^{(1)} \circ (\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})$

$= \partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$ from(1.2) we have

$= \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{31}^{(1)} \circ \partial_{32}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$

$= \partial_{21}^{(1)} \circ \partial_{32}^{(2)}$

$= k_3 \circ P$ □

Finally ,we can define the maps $\sigma_1, \sigma_2\ and\ \sigma_3$ where:

$$\sigma_{3:} D_8F\otimes D_5F\otimes D_1F \longrightarrow \begin{array}{c} D_8F\otimes D_4F\otimes D_2F \\ \oplus \\ D_7F\otimes D_6F\otimes D_1F \end{array}$$

$$\sigma_2 : \begin{array}{c} D_8F \otimes D_4F \otimes D_2F \\ \oplus \\ D_7F \otimes D_6F \otimes D_1F \end{array} \longrightarrow \begin{array}{c} D_7F \otimes D_4F \otimes D_3 \\ \oplus \\ D_6F \otimes D_6F \otimes D_2F \end{array}$$

and

$$\sigma_1 : \begin{array}{c} D_7F \otimes D_4F \otimes D_3F \\ \oplus \\ D_6F \otimes D_6F \otimes D_2F \end{array} \longrightarrow D_6F \otimes D_5F \otimes D_3$$

by • $\sigma_3(x) = \big(n_1(x), k_1(x)\big)$; $\forall x \in D_8F \otimes D_5F \otimes D_1F$

• $\mathrm{\sigma_2\big((x_1, x_2)\big) = \big(n_2(x) - p(x_2)\big), m_1(x_2) - k_2(x_1)}$;

$$\forall (x_1, x_2) \in \begin{array}{c} D_8F \otimes D_4F \otimes D_2F \\ \oplus \\ D_7F \otimes D_6F \otimes D_1F \end{array}$$

•$\sigma_1\big((x_1, x_2)\big) = \big(k_3(x_1) + m_2(x_2)\big)$;

$$\forall (x_1, x_2) \in \begin{array}{c} D_7F \otimes D_4F \otimes D_3F \\ \oplus \\ D_6F \otimes D_6F \otimes D_2F \end{array}$$ □

Proposition 3.

$$0 \rightarrow D_8F\otimes D_5F\otimes D_1F \overset{\sigma_3}{\rightarrow} \begin{array}{c} D_8F\otimes D_4F\otimes D_2F \\ \oplus \\ D_7F\otimes D_6F\otimes D_1F \end{array} \overset{\sigma_2}{\rightarrow} \begin{array}{c} D_7F\otimes D_4F\otimes D_3F \\ \oplus \\ D_6F\otimes D_6F\otimes D_2F \end{array} \overset{\sigma_1}{\rightarrow} D_6F\otimes D_5F\otimes D_3F$$

Is complex

Proof :

From the definition we know that $\partial_{32}^{(1)}$ and$\partial_{21}^{(1)}$ are injective [2],then

We get σ_3 is injective. Now

$\sigma_2 \circ \sigma_3(x) = \sigma_2(n_1(x), k_1(x)) = \sigma_2(\partial_{32}^{(1)}(x), \partial_{21}^{(1)}(x))$

$= (n_2(\partial_{32}^{(1)}(x)) - P(\partial_{21}^{(1)}(x)), m_1(\partial_{21}^{(1)}(x)) - k_2(\partial_{32}^{(1)}(x)))$

Now

$n_2(\partial_{32}^{(1)}(x)) - P(\partial_{21}^{(1)}(x)) = (\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)}) \circ \partial_{32}^{(1)}(x) - \partial_{32}^{(2)} \circ \partial_{21}^{(1)}(x)$

$=(\partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{31}^{(1)} \circ \partial_{32}^{(1)} - \partial_{32}^{(2)} \circ \partial_{21}^{(1)})(x)$

$=(\partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(2)} \circ \partial_{21}^{(1)})(x)$

$=0$

and

$m_1(\partial_{21}^{(1)}(x)) - k_2(\partial_{32}^{(1)}(x)) = (\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)}) \circ \partial_{21}^{(1)}(x) - \partial_{21}^{(2)} \circ \partial_{32}^{(1)}(x)$

$= (\partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{31}^{(1)} \circ \partial_{21}^{(1)} - \partial_{21}^{(2)} \circ \partial_{32}^{(1)})(x)$

$=(\partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(2)} \circ \partial_{32}^{(1)})(x)$

$= 0$

So we have $(\sigma_2 \circ \sigma_3)(x) = 0$

and

$(\sigma_1 \circ \sigma_2)(x_1, x_2) = \sigma_1((n_2(x_1) - P(x_2)), m_1(x_2) - k_2(x_1))$

$=\sigma_1((\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})(x_1) - \partial_{32}^{(2)}(x_2), (\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})(x_2)\partial_{21}^{(2)}(x_1))$

$=\partial_{21}^{(1)}(\frac{1}{2}\partial_{21}^{(1)} \circ \partial_{32}^{(1)} + \partial_{31}^{(1)})(x_1) - \partial_{21}^{(1)} \circ \partial_{32}^{(2)}(x_2) + \partial_{32}^{(1)}(\frac{1}{2}\partial_{32}^{(1)} \circ \partial_{21}^{(1)} - \partial_{31}^{(1)})(x_2)$

$-\partial_{32}^{(1)} \circ \partial_{21}^{(2)}(x_1)$ $=(\partial_{21}^{(2)} \circ \partial_{32}^{(1)} + \partial_{21}^{(1)} \circ \partial_{31}^{(1)} - \partial_{32}^{(1)} \circ \partial_{21}^{(2)})(x_1) +$

$(\partial_{32}^{(2)} \circ \partial_{21}^{(1)} - \partial_{32}^{(1)} \circ \partial_{31}^{(1)} - \partial_{21}^{(1)} \circ \partial_{32}^{(2)})(x_2)$

But

$\partial_{21}^{(2)} \circ \partial_{32}^{(1)} = \partial_{32}^{(1)} \circ \partial_{21}^{(2)} - \partial_{21}^{(1)} \circ \partial_{31}^{(1)}\partial_{32}^{(2)} \circ \partial_{21}^{(1)} = \partial_{21}^{(1)} \circ \partial_{32}^{(2)} + \partial_{32}^{(1)} \circ \partial_{31}^{(1)}$

Which implies that $(\sigma_1 \circ \sigma_2)(x_1, x_2) = 0$ □.

ACKNOWLEDGMENT

Thanks To The auther with the support I received from Al-MustansiriyaUniversity ,Iraq .

REFERENCES

[1] K Akin, D. A. Buchsbaum and J .Weyman ,Schurfunctors And Schur complex ,Adv .Math .44,207-278 (1982) .
[2] D. A. Buchsbaum and G. C. Rota ,Approaches to resolution of Weyl Modules, Adv. In applied math.27,82-191 (2001).
[3] G . Boffi and D. A. Buchsbaum , Threading homology through algebra :Selected Patterns ,clarendon press .oxford.
[4] D. A Buchsbaum and G.C. Rota ,A new construction in homological Algebra ,Natl .a cad .Sci ,USA 91 ,4115-4119 (1994).
[5] D. A Buchsbaum ,Characteristic free example of Lascoux resolution and Letter place methods for intertwining numbers, European Journal of Gombinatorics ,Vol. 25, 1169-1179 (2004).
[6] H. R. Hassan, Application of the characteristic-free resolution of Weyl Module to the Lascoux resolution in the case (3,3,3) ph. D thesis, universita di Roma "Tor Vergata" (2006).
[7] H. R. Hassan, The Reduction of Resolution of Weyl Module From characteristic -free Resolution in case (4,4,3), J, Ibn Al –HaithamFor pure and applied Science, 25, 341-355(2012) .
[8] J. J. Rotman ,Introduction to homological algebra , A cademic press.INC(1979).