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GENERALIZED FRACTIONALCALCULUS OF THE GENERALIZED HURWITZ-LERCH ZETA FUNCTION

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Abstract*—This paper deals with the derivation of the generalized fractional differentiation and generalized fractional integration of the generalized Hurwitz-Lerch Zeta function defined and studied by Srivastava et al [16]. Representations of such relations are obtained in terms of Riemann-Lioville integrals.

Mathematics Subject Classification: - 33C60, 26A33.

Index Terms- Generalized Hurwitz-Lerch Zeta function, Riemann-Liouville fractional integrals, Saigo fractional integrals.

INTRODUCTION

This paper is devoted to the investigation of the generalized Hurwitz-Lerch Zeta function defined by Srivastava et al [16]

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}}\frac{z^n}{(n+a)^s\, n!} \tag{1}$$

$(\lambda,\mu \in \mathbb{C};\ \nu \in \mathbb{C}\setminus \mathbb{Z}_0^-;\ \rho,\sigma,\kappa \in \mathbb{R}^+;$
$\kappa-\rho-\sigma > -1$ when $s,z \in \mathbb{C};$
$\kappa-\rho-\sigma = -1$ and $s \in \mathbb{C}$
when $|z| < \delta^* := \rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa};$
$\kappa-\rho-\sigma = -1$ and $\Re(s+\nu-\lambda-\mu) > 1$
when $|z| = \delta^*$).

$(\lambda)_n$ is the Pochhammer symbol for

$\lambda,\ n \in \mathbb{C}$ is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$$

$$= \begin{cases} 1 & (n=0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N};\ \lambda\in\mathbb{C}) \end{cases}$$

it being understood conventionally that $(0)_0 = 1$.

For $\lambda = \rho = 1$, (1) yield the generalized Hurwitz-Lerch Zeta function defined by Lin and Srivastava [6].

$$\Phi_{1,\mu;\nu}^{(1,\sigma,\kappa)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\sigma n}}{(\nu)_{\kappa n}}\frac{z^n}{(n+a)^s} = \Phi_{\mu;\nu}^{(\sigma,\kappa)}(z,s,a) \tag{2}$$

Further we set $\rho=\sigma=\kappa=1$, (1) yield the generalized Hurwitz-Lerch Zeta function studied by Garg et al [3] and Jankov et al [5] as follows:.

$$\Phi_{\lambda,\mu;\nu}^{(1,1,1)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n}\frac{z^n}{(n+a)^s\, n!} = \Phi_{\lambda,\mu;\nu}(z,s,a). \tag{3}$$

when $\rho=\sigma=\kappa=1$ and $\lambda=\nu$, (1) yield the Hurwitz-Lerch Zeta function studied by

Goyal and Laddha [4, p. 100, Equation (1.5)] as detailedbelow

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$$ {}^{(1,1,1)}_{n,m,n}(z,s,a) \qquad \underset{n\ \ 0}{} \frac{m_{\ s\,n}\, z^{n}}{n \quad a^{\ s}\, n!} \qquad {}^{*}_{m}(z,s,a). \tag{4}$$

Lemma 1 [16, eq. 3.5, p 498] the generalized Hurwitz-Lerch Zeta function in term of $\overline{H}$ - function

$$ {}^{(r,s,k)}_{l,m,n}(z,s,a) \quad \frac{(n)}{(l\ )\quad (m)} \overline{H}^{1,3}_{3,3} \quad z \left| \begin{matrix} (1 \quad l\ , r;1),(1 \quad m,s\ ;1),(1 \quad a,1;s) \\ (0,1),(1 \quad n,k;1),(\quad a,1;s) \end{matrix} \right. . \tag{5}$$

For further details about Hurwitz-Lerch Zeta function the reader is referres to the Srivasatava [13], Srivasatava et al [14] and Saxena et al [10]

FRACTIONAL CALCULUS OPERATORS

For $\alpha \in C\ (R(\alpha) > 0)$ the Riemann-Liouville left and right-sided fractional calculus operators are defined as follows (See [9])

$$\left(I^{\alpha}_{0+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt \quad , \quad x > 0 \tag{6}$$

$$\left(I^{\alpha}_{-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}}\, dt \quad , \quad x > 0 \tag{7}$$

and

$$\left(D^{\alpha}_{0+} f\right)(x) = \left(\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1]} \left(I^{1-\alpha+[\mathrm{Re}(\alpha)]}_{0+} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1]} \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha-[\mathrm{Re}(\alpha)]}}\, dt \quad , \quad x > 0 \tag{8}$$

$$\left(D^{\alpha}_{-} f\right)(x) = \left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1]} \left(I^{1-\alpha+[\mathrm{Re}(\alpha)]}_{-} f\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1]} \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])} \int_x^{\infty} \frac{f(t)}{(t-x)^{\alpha-[\mathrm{Re}(\alpha)]}}\, dt \quad , \quad x > 0 \tag{9}$$

where $[\mathrm{Re}(\alpha)]$ is the integral part of $\mathrm{Re}(\alpha)$.

For $\alpha, \beta, \eta \in C$ and $x > 0$ the generalized fractional calculus operators given by Saigo [8] are defined as

$$\left(I^{\alpha,\beta,\eta}_{0+} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta, -\eta\ ; \alpha\ ; 1-\frac{t}{x}\right) f(t)\, dt \ \left[\mathrm{Re}(\alpha) > 0\right] \tag{10}$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{\alpha+n,\beta-n,\eta-n} f\right)(x) \qquad \left(\operatorname{Re}(\alpha) < 0\ ;\ \eta = [\operatorname{Re}(-\alpha)] + 1\right)$$

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty t^{-\alpha-\beta} (t-x)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta, -\eta\ ;\alpha\ ;\ 1-\frac{x}{t}\right) f(t)\,dt \qquad [\operatorname{Re}(\alpha) > 0]$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{0+}^{\alpha+n,\beta-n,\eta} f\right)(x) \quad \left(\operatorname{Re}(\alpha) < 0\ ;\ \eta = \left[\operatorname{Re}(-\alpha)\right] + 1\right) \quad (11)$$

$$\left(D_{0+}^{\alpha,\beta,\eta} f\right)(x) \equiv \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta}\right) f(x) \qquad [\operatorname{Re}(\alpha) > 0]$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n} f\right)(x) \quad \left(\operatorname{Re}(\alpha) > 0\ ;\ \eta = \left[\operatorname{Re}(\alpha)\right] + 1\right) \quad (12)$$

$$\left(D_{-}^{\alpha,\beta,\eta} f\right)(x) = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta}\right) f(x) \qquad [\operatorname{Re}(\alpha) > 0]$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta} f\right)(x) \quad \left(\operatorname{Re}(\alpha) > 0\ ;\ \eta = \left[\operatorname{Re}(\alpha)\right] + 1\right) \quad (13)$$

${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric function defined for complex parameters $a, b, c, \in C \quad (c \neq 0, -1, \ldots)$ by the hypergeometric series

$${}_2F_1(a, b\ ;c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_k\ (b)_k}{(c)_k} \frac{z^n}{n!} \qquad (14)$$

with converges absolutely for $|z| < 1$ and for $|z| = 1$, it Converges for $R\ (c-a-b) > 0$; (see [1,2]). For a detailed account of hypergeometric function. [See (7, 12)].

Note 3: Inequalities associated with Čebyšev functional for Saigo fractional integration operator defined are given by Saxena et al [11].

The generalized fractional calculus operator (15), (16), (17) and (18) coincide
if $\beta = -\alpha$ with the Riemann-Liouville operators (6) – (9) for $\operatorname{Re}(\alpha) > 0$:

$$\left(I_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0+}^{\alpha} f\right)(x), \qquad (15)$$

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x), \qquad (16)$$

$$\left(D_{0+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{0+}^{\alpha} f\right)(x), \qquad (17)$$

$$\left(D_{0-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(D_{0-}^{\alpha} f\right)(x), \qquad (18)$$

3. RESULTS REQUIRED IN THE SEQUEL

Lemma 1. we have [9]

$$\left(I_{0+}^{\alpha,\beta,\eta}\ t^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)\,\Gamma(\lambda+\eta-\beta)}{\Gamma(\lambda-\beta)\Gamma(\lambda+\alpha+\eta)}\,x^{\lambda-\beta-1} \tag{19}$$

where $\alpha,\beta,\eta,\lambda\in C$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\lambda)>\max[0,\mathrm{Re}(\beta-\eta)]$

Lemma 2. we have [9]

$$\left(I_{-}^{\alpha,\beta,\eta}\ t^{-\lambda}\right)(x)=\frac{\Gamma(\beta+\lambda)\,\Gamma(\eta+\lambda)}{\Gamma(\lambda)\Gamma(\alpha+\beta+\eta+\lambda)}\,x^{-\lambda-\beta} \tag{20}$$

where $\alpha,\beta,\eta,\lambda\in C$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\lambda)<\min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]$.

Lemma 3. Saxena et al [13] introduced following form

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\alpha-\beta)\Gamma(\rho+\alpha+\alpha'+\beta'-\gamma)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\alpha'-\gamma)\Gamma(\rho+\alpha+\beta'-\gamma)}\,x^{\rho+\alpha+\alpha'-\gamma-1} \tag{21}$$

where Re(r) 0, Re(r a b) 0, Re(r a a b g) 0, (x 0).

Lemma 3. Saxena et al [13] introduced following form

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{\rho-1}\right)(x)=(-1)^{n}\,\frac{\Gamma(1-\alpha-\alpha'+\gamma-\rho)\Gamma(1-\alpha'-\beta+\gamma-\rho)\Gamma(1+\beta'-\rho)}{\Gamma(1-\rho)\Gamma(1-\alpha-\alpha'-\beta+\gamma-\rho)\Gamma(1-\alpha'+\beta'-\rho)}\,x^{\rho+\alpha+\alpha'-\gamma-1} \tag{22}$$

where Re(1 b r) 0, Re(1 r a b g) 0, Re(1 r a a g) 0, (x 0).

Theorem 1: Let l , m ; n \ $_0$; r , s , k , further let $\alpha,\beta,\eta\in C$ with Re $(\alpha)>0$, $\xi>0$,Re $(\beta)>0$,a $\in$ C, $\mathrm{Re}(\gamma)>\max[0,\mathrm{Re}(\beta-\eta)]$. For $x>0$ there holds the formula

$$I_{0+}^{\alpha,\beta,\eta}\left[t^{\lambda-1}\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),(1-\gamma+\beta-\eta,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma+\beta,\xi;1),(1-\gamma-\alpha-\eta,\xi;1)(-a,1;s)\end{matrix}\right.\right]. \tag{23}$$

Proof :- Using equation (1) , (5), (10) and (19), we have

$$I_{0+}^{\alpha,\beta,\eta}\left[t^{\gamma-1}\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(bt^{\xi},s,a)\right]=I_{0+}^{\alpha,\beta,\eta}\left(t^{\gamma-1}\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}(n+a)^{s}}\frac{(bt^{\xi})^{n}}{n!}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}(n+a)^{s}}\frac{b^{\xi}}{n!}\; I_{0+}^{\alpha,\beta,\eta}\left[t^{\xi n+\lambda-1}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}(n+a)^{s}}\frac{b^{\xi}}{n!}\;\frac{\Gamma(\xi n+\lambda)\Gamma(\xi n+\lambda-\beta+\eta)}{\Gamma(\xi n+\lambda-\beta)\Gamma(\xi n+\lambda+\alpha+\eta)}x^{\xi n+\lambda-\beta-1}$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+\rho n)\,\Gamma(\mu+\sigma n)\,\Gamma(\nu)\;\Gamma(\xi n+\gamma)\Gamma(\xi n+\gamma-\beta+\eta)}{\Gamma(\lambda)\Gamma(\mu)\Gamma(\nu+kn)\Gamma(\xi n+\gamma-\beta)\Gamma(\xi n+\gamma+\alpha+\eta)(n+a)^{s}}\frac{b^{\xi}}{n!}x^{\xi n+\lambda-\beta-1}$$

$$=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+\rho n)\,\Gamma(\mu+\sigma n)\;\;\Gamma(\xi n+\gamma)\Gamma(\xi n+\gamma-\beta+\eta)}{\Gamma(\nu+kn)\Gamma(\xi n+\gamma-\beta)\Gamma(\xi n+\gamma+\alpha+\eta)(n+a)^{s}}\frac{b^{\xi}}{n!}x^{\xi n}$$

$$=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),(1-\gamma+\beta-\eta,\xi;1),\,(1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma+\beta,\xi;1),(1-\gamma-\alpha-\eta,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

Theorem 1 is thus proved.

For $\beta=-\alpha$ in Theorem 1, it gives the result in terms of Riemann-Lioville fractional integral operator as follows:

Corollary 1.1

$$I_{0+}^{\alpha}\left[t^{\lambda-1}\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{4,4}^{1,4}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),(1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma+\beta,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

For $\lambda=\rho=1$ in Theorem 1, it gives the following Corollary in terms of generalized Hurwitz-Lerch Zeta function

Corollary 1.2

$$I_{0+}^{\alpha,\beta,\eta}\left[t^{\lambda-1}\,\Phi_{\mu,\nu}^{\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1,1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),(1-\gamma+\beta-\eta,\xi;1),\,(1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma+\beta,\xi;1),(1-\gamma-\alpha-\eta,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

Theorem 2: Let l , m ; n \ $_{0}$; r , s , $\mathbf{k}$, further let $\alpha,\beta,\eta\in C$ with $\mathrm{Re}(\alpha)>0$, $\xi>0$, $\mathrm{Re}(\beta)>0$, $a\in C$, $\mathrm{Re}(\gamma)<\min[\mathrm{Re}(\beta),\mathrm{Re}(\eta)]$. For $x>0$ there holds the formula

$$I_{0-}^{\alpha,\beta,\eta}\left[t^{\lambda-1}\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(\gamma-\beta,-\xi;1),(\gamma-\eta,-\xi;1),\,(1-a,1;s)\\(0,1),(1-\nu,k;1),(\gamma,-\xi;1),(\gamma-\alpha-\beta-\eta,-\xi;1),(-a,1;s)\end{matrix}\right.\right]. \quad (24)$$

The proof can be developed on similar lines to that of Theorem 1 .

For $\beta=-\alpha$ in Theorem 2, it gives the result in terms of Riemann-Lioville fractional integral operator as follows :

Corollary 2.1

$$I_{0-}^{\alpha}\left[t^{\lambda-1}\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{4,4}^{1,4}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(\gamma+\alpha,-\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(\gamma,-\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

For $\lambda=\rho=1$ in Theorem 2, it gives the following Corollary in terms of generalized Hurwitz-Lerch Zeta function

Corollary 2.2

$$I_{0-}^{\alpha,\beta,\eta}\left[t^{\lambda-1}\,\Phi_{\mu,\nu}^{\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1,1),(1-\mu,\sigma;1),(\gamma-\beta,-\xi;1),(\gamma-\eta,-\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(\gamma,-\xi;1),(\gamma-\alpha-\beta-\eta,-\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

The following two Theorems can be proved in a similar manner.

Theorem 3: Let $l\ ,m\quad ;\ n\quad \backslash\ {}_0\ ;r\ ,s\ ,\mathrm{k}$, further let $\alpha,\beta,\eta\in C$ with $\operatorname{Re}(\alpha)>0$, $\xi>0$, $\operatorname{Re}(\beta)>0$, $a\in C$, $\eta=\left[\operatorname{Re}(\alpha)\right]+1$, $\operatorname{Re}(\gamma)>\max\left[0,\operatorname{Re}(\beta-\eta)\right]$. For $x>0$ there holds the formula

$$D_{0+}^{\alpha,\beta,\eta}\left[t^{\lambda-1}\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,k)}(bt^{\xi},s,a)\right]=\frac{x^{\gamma+\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),(1-\gamma-\alpha-\beta-\eta,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma-\beta,\xi;1),(1-\gamma-\eta,\xi;1),(-a,1;s)\end{matrix}\right.\right]. \quad (25)$$

For $\beta=-\alpha$ in Theorem 3, it gives the result in terms of Riemann-Lioville fractional integral operator as follows :

Corollary 3.1

$$D_{0+}^{\alpha}\left[t^{\lambda-1}\,\Phi_{\lambda,\mu,\nu}^{\rho,\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\alpha-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{4,4}^{1,4}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1+\alpha-\gamma,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

For $\lambda=\rho=1$ in Theorem 3, it gives the following Corollary in terms of generalized Hurwitz-Lerch Zeta function

Corollary 3.2

$$D_{0+}^{\alpha,\beta,\eta}\left[t^{\lambda-1}\,\Phi_{\mu,\nu}^{\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma+\beta-1}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1,1),(1-\mu,\sigma;1),(1-\gamma,\xi;1),(1-\gamma-\alpha-\beta-\eta,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma-\beta,\xi;1),(1-\gamma-\eta,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

Theorem 4: Let $l\ ,m\quad ;\ n\quad \backslash\ {}_0\ ;r\ ,s\ ,\mathrm{k}$, further let $\alpha,\beta,\eta\in C$ with $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\beta)>0$, $\xi>0$, $a\in C$, $\eta=\left[\operatorname{Re}(\alpha)\right]+1$, $\operatorname{Re}(\gamma)<\min\left[\operatorname{Re}(\beta),\operatorname{Re}(\eta)\right]$ For $x>0$ there holds the formula

$$D_{0-}^{\alpha,\beta,\eta}\left[t^{-\gamma}\,\Phi_{\lambda,\mu,\nu}^{\rho,\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma+\beta}\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma+\beta,\xi;1),(1-\gamma-\alpha-\eta,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma,\xi;1),(1+\beta-\gamma-\eta,\xi;1),(-a,1;s)\end{matrix}\right.\right]. \quad (26)$$

For $\beta=-\alpha$ in Theorem 3, it gives the result in terms of Riemann-Lioville fractional integral operator as follows :

Corollary 4.1

$$D_{0-}^{\alpha}\left[t^{-\gamma}\ \Phi_{\lambda,\mu,\nu}^{\rho,\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma-\alpha}\,\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{4,4}^{1,4}\left[-bx^{\xi}\left|\begin{matrix}(1-\lambda,\rho;1),(1-\mu,\sigma;1),(1-\gamma-\alpha,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

For $\lambda=\rho=1$ in Theorem 3, it gives the following Corollary in terms of generalized Hurwitz-Lerch Zeta function

Corollary 4.2

$$D_{0-}^{\alpha,\beta,\eta}\left[t^{-\lambda}\ \Phi_{\mu,\nu}^{\sigma,k}(bt^{\xi},s,a)\right]=\frac{x^{\gamma+\beta}\,\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\bar{H}_{5,5}^{1,5}\left[-bx^{\xi}\left|\begin{matrix}(1,1),(1-\mu,\sigma;1),(1-\gamma+\beta,\xi;1),(1-\gamma-\alpha-\eta,\xi;1),\ (1-a,1;s)\\(0,1),(1-\nu,k;1),(1-\gamma,\xi;1),(1+\beta-\gamma-\eta,\xi;1),(-a,1;s)\end{matrix}\right.\right].$$

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