International Journal of Engineering Research And Management (IJERM)
ISSN : 2349- 2058, Volume-03, Issue-04, April 2016

Formulations on the Advanced Notions in Analytical Dynamics of Mechanical Systems

Iuliu Negrean, Kalman Kacso

***Abstract* — This paper is devoted to the presentation of formulations on the advanced notions that are used in the analytical dynamics study of the mechanical systems. In the advanced notions the acceleration energies of higher order, also named "kinetic energies of the accelerations of higher order" are included. In Newtonian dynamics studies, the kinetic energy is used as a central function in Lagrange - Euler equations. Using the main author's researches, this paper extends the study by developing the acceleration energies of first, second and third order. The analysis, performed in this paper, highlights the importance of acceleration energies of higher order in the study of dynamic behavior of mechanical systems with suddenly movements, as well as in the transient phases of motion. They will be implemented in the differential equations of motion of higher order (at least two). In these situations, the higher order time variations of the linear and angular accelerations are developed. Integral part of the mechanical systems is the mechanical structure of the serial robots. As a result, in the final part of this paper, an application on a serial robot that emphasizes an essential aspect regarding the time variation law of the acceleration energies of higher order, as well as their importance in the dynamic behavior will be presented.**

***Index Terms*—Acceleration energies, analytical dynamics, multibody systems, robotics.**

I. Introduction

This paper is structured in two main parts. The first one is focused on a few formulations, in the analytical dynamics of multibody systems (MBS), when they are characterized by suddenly movements (when the linear acceleration greater then g - gravitational acceleration), and the transitory motions. It demonstrates theoretical and experimental the existing of the acceleration energy of higher order. On the basis of the main author's researches the acceleration energies of first, second and third orders will be presented in both explicit and matrix form. They will be integrated into differential equations of motion in higher order, which will lead to variations in time of generalized forces, which dominating these types of mechanical systems. The last part is an application in which the theoretical aspects presented in this paper are used to obtain the acceleration energies of higher order for a serial robot of type Fanuc.

Manuscript received April 23 , 2016.
Iuliu Negrean, Department of Mechanical Systems Engineering, Technical University of Cluj-Napoca, Cluj-Napoca, Romania
Kalman Kacso, Department of Mechanical Systems Engineering, Technical University of Cluj-Napoca, Cluj-Napoca, Romania

II. Advanced Notions in Analytical Dynamics

The phrase, entitled "advanced notions" founded in the analytical dynamics, is focused in this paper on the energies whose central functions are referring to the accelerations of higher order. They are developing in any suddenly and transitory motion of the mechanical systems. Leading to Appell's function, highlighted in 1899, also named "the kinetic energy of the accelerations" [9], the main author has been developed a few mathematical formulations on the expressions for acceleration energies of first, second and third order [4], [6],]7]. They will be presented, in explicit and matrix form, and the kinematical parameters will be expressed, using matrix exponential functions [3], [5], [9].

The starting equation is written in the generalized form as:

$$\begin{aligned} E_A^{(p)i} &= \frac{1}{2}\int \overset{(p)}{\overline{v}}{}_i^T \cdot \overset{(p)}{\overline{v}}_i \cdot dm = \frac{1}{2}\int Trace\left(\overset{(p+1)}{\overline{r}_i} \cdot \overset{(p+1)}{\overline{r}}{}_i^T\right)\cdot dm = \\ &= \frac{1}{2}\cdot Trace\left(\overset{(p+1)}{\overline{r}_{C_i}} \cdot \overset{(p+1)}{\overline{r}}{}_{C_i}^T\right)\int dm + \\ &+ \frac{1}{2}\cdot Trace\left[\overset{(p+1)}{\overline{r}_{C_i}} \cdot \int {}^i\overline{r}_i^{*T} \cdot dm \cdot {}_i^0\overset{(p+1)}{[R]}{}^T\right] + \\ &+ \frac{1}{2}\cdot Trace\left[{}_i^0\overset{(p+1)}{[R]} \cdot \int {}^i\overline{r}_i^{*T} \cdot dm \cdot \overset{(p+1)}{\overline{r}}{}_{C_i}^T\right] + \\ & \frac{1}{2}\cdot Trace\left[\int {}_i^0\overset{(p+1)}{[R]} \cdot {}^i\overline{r}_i^{*} \cdot {}^i\overline{r}_i^{*T} \cdot {}_i^0\overset{(p+1)}{[R]}{}^T \cdot dm\right] \end{aligned} \quad (1)$$

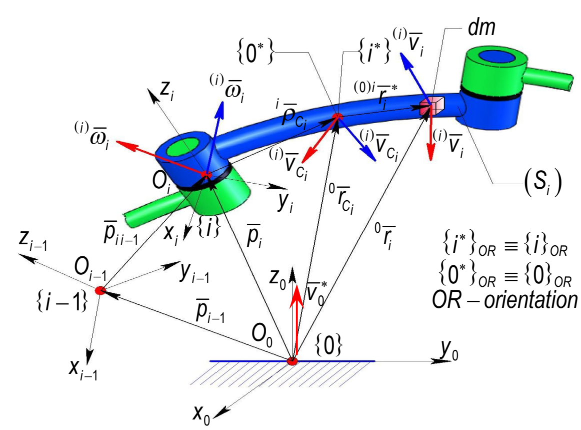


Fig. 1 A kinetic ensemble from MBS

$$E_A^{(p)}\left[\overline{\theta}(t);\overline{\theta}(t);\quad;\overset{(p+1)}{\overline{\theta}}(t)\right]=$$
$$=\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[\int {}^i\overline{r}_i^{*}\cdot{}^i\overline{r}_i^{*T}\cdot dm+\right.\right.$$
$$\left.\left.+{}^i\overline{r}_{C_i}\cdot{}^i\overline{r}_{C_i}^{T}\cdot\int dm\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\} \tag{2}$$
$$+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\overline{p}_i}\cdot\overset{(p+1)}{\overline{p}_i^T}\right]\cdot\int dm=$$
$$\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{\overset{(p+1)}{{}_i^0[R]}\cdot\left[{}^iI_{pi}^{*}+M_i\cdot{}^i\overline{r}_{C_i}\cdot{}^i\overline{r}_{C_i}^{T}\right]\cdot\overset{(p+1)}{{}_i^0[R]^T}\right\}+$$
$$+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overset{(p+1)}{\overline{p}_i}\cdot\overset{(p+1)}{\overline{p}_i^T}\right]\cdot M_i$$

Considering the symbols from “Fig.1”, refer to “(1)” it defines the acceleration energy of order $"p=1,2,3,..."$ for a kinetic ensemble $"i=1\rightarrow n"$, belonging to MBS, while refer to “(2),” it is corresponding to whole mechanical system. The symbols, included in (1), (2) and “Fig.1” have the significances: $"(p)"$ and $"(p+1)"$ is the order of the absolute and time derivatives; ${}^i\overline{r}_i^{*}$ the position vector of the elementary mass $"dm"$ relative to reference frame $\{i^{*}\}$ applied in the mass center; ${}^{(i)}\overline{r}_{C_i}$ the position vector of the mass center projected on $\{0\}$ or $\{i\}$ reference frame; ${}_i^0[R]$ the rotation matrix between the two frames above mentioned. Whereas MBS is characterized by $"n"$ degrees of freedom (generalized coordinates), they are included in the column matrix $\overline{\theta}(t)=(q_i(t),\ for\ i=1\rightarrow n)^T$, as well their time derivatives. “Using (1) and (2), in the following of the first chapter, the expressions of for acceleration energies of first, second and third orders will be presented, in explicit and matrix form”.

A. Acceleration Energy of First Order

According to the scientific literature [1], [2], [6]-[10], in 1879, Gibbs defines the differential equations of motion, on which, in 1899, Paul Appell performs a detailed study. As a result of this study were deduced the equations known as Gibbs-Appell which are applied for holonomic and nonholonomic systems, where the role of the kinetic energy was substituted by the acceleration energy also known as Appell’s function or “kinetic energy of acceleration” [9]. Unlike the studies, above mentioned, in the paper [4] and not only, the main author was established the acceleration energy in a generalized form, corresponding to a MBS and it was named acceleration energy of first order. “Starting from (1) and (2), equation for defining the acceleration energy of first order is”:

$$E_A^{(1)i}=\frac{1}{2}\cdot\int\overline{v}_i^T\cdot\overline{v}_i\cdot dm=\frac{1}{2}\cdot\int Trace\left(\overline{r}_i\cdot\overline{r}_i^T\right)\cdot dm=$$
$$=\frac{1}{2}\cdot\int Trace\left[\left(\overline{r}_{C_i}+{}_i^0[R]\cdot{}^i\overline{r}_i^{*}\right)\cdot\left(\overline{r}_{C_i}^T+{}^i\overline{r}_i^{*T}\cdot{}_i^0[R]^T\right)\right]\cdot dm \tag{3}$$

After significant matrix and differential transformations in (1) and (2), the main author was obtained the expression of definition, in explicit form, for acceleration energy of first order and corresponding to whole mechanical system (MBS):

$$E_A^{(1)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right]=$$
$$=(-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\sum_{i=1}^{n}\left[\frac{1}{2}\cdot M_i\cdot{}^{(i)}\overline{v}_{C_i}^T\cdot{}^{(i)}\overline{v}_{C_i}\right]+$$
$$+\Delta_m^2\cdot\sum_{i=1}^{n}\frac{1}{2}\cdot{}^{(i)}\overline{\omega}_i^T\cdot{}^{(i)}I_i^{*}\cdot{}^{(i)}\overline{\omega}_i+$$
$$+\Delta_m^2\cdot\sum_{i=1}^{n}\left[{}^{(i)}\overline{\omega}_i^T\cdot\left({}^{(i)}\overline{\omega}_i\times{}^{(i)}I_i^{*}\cdot{}^{(i)}\overline{\omega}_i\right)\right]+ \tag{4}$$
$$+\Delta_m^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^{(i)}\overline{\omega}_i^T\cdot\left[{}^{(i)}\overline{\omega}_i^T\cdot Trace\left({}^{(i)}I_{pi}^{*}\right)\cdot{}^{(i)}\overline{\omega}_i-\right.\right.$$
$$\left.\left.-{}^{(i)}\overline{\omega}_i^T\cdot{}^{(i)}I_{pi}^{*}\cdot{}^{(i)}\overline{\omega}_i\right]\cdot{}^{(i)}\overline{\omega}_i\right\}$$

Refer to “(4),” it includes the symbols and parameters:

$$\Delta_M=\left\{\{-1;general\ motion\};\{0;translation\};\{1;rotation\}\right\};$$

${}^{(i)}\overline{v}_{C_i}$ the absolute acceleration of the mass center; ${}^{(i)}\overline{\omega}_i$ and ${}^{(i)}\overline{\omega}_i$ the absolute angular velocity and acceleration of the kinetic ensemble (i) from “Fig.1”; M_i, ${}^{(i)}I_i^{*}$ and ${}^{(i)}I_{pi}^{*}$ represent the mass, the axial and centrifugal inertial tensor, as well as the planar centrifugal inertial tensor corresponding to the entire kinetic ensemble (i), relative to the mass center C_i:

$${}^{(i)}I_i^{*}=\int\left({}^{(i)}\overline{r}_i^{*}\times\right)\left({}^{(i)}\overline{r}_i^{*}\times\right)^T dm,\ {}^{(i)}I_{pi}^{*}=\int{}^{(i)}\overline{r}_i^{*}\cdot{}^{(i)}\overline{r}_i^{*T}dm. \tag{5}$$

The same acceleration energy of first order, above written by (4) and corresponding to the MBS, the main author has developed [4], [6] a matrix expression, and below presented:

$$E_A^{(1)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right]=\frac{1}{2}\cdot\left\{\overline{\theta}^T(t)\cdot M\left[\overline{\theta}(t)\right]\cdot\overline{\theta}(t)+\right.$$
$$+\overline{\theta}^T(t)\cdot V\left[\overline{\theta}(t);\overline{\theta}^2(t)\right]+ \tag{6}$$
$$\left.+\left[\overline{\theta}^T(t)\cdot D\left[\overline{\theta}(t);\overline{\theta}^2(t)\right]\cdot\overline{\theta}(t)\right]\right\}.$$

Refer to “(6),” it contains on the one hand the first and second time derivatives of the column matrix $\overline{\theta}(t)$ of the generalized variables , also named generalized velocities and accelerations. On the other hand in (6) a set of dynamics matrices are founded, whose expressions are defined below:

$$M\left(\overline{\theta}\right)=\underset{(n\times n)}{Matrix}\left\{M_{ij}=M_{ji}\quad\begin{matrix}i=1\rightarrow n\\ j=1\rightarrow n\end{matrix}\right\} \tag{7}$$

where $M_{ij}=\sum_{k=\max(i;j)}^{n}Trace\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kj}^T\right]$ (8)

$$V\left(\overline{\theta};\overline{\theta}^2\right)=\underset{(n\times 1)}{Matrix}\left(V_i,\ i=1\rightarrow n\right) \tag{9}$$

where $V_i=\overline{\theta}^T\cdot\left[\left\{V_{ijm}=V_{imj}\right\}\quad\begin{matrix}j=1\rightarrow n\\ m=1\rightarrow n\end{matrix}\right]\cdot\overline{\theta}$ (10)

and $V_{ijm}=\sum_{k=\max(i;j)}^{n}Trace\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjm}^T\right]$ (11)

International Journal of Engineering Research And Management (IJERM)
ISSN : 2349- 2058, Volume-03, Issue-04, April 2016

$$D\left(\bar{\theta};\bar{\theta}^2\right)=\underset{(n\times n)}{Matrix}\left\{D_{ij} \quad \begin{matrix} i=1\to n \\ j=1\to n \end{matrix}\right\} \tag{12}$$

where $$D_{ij}=\bar{\theta}^T\cdot\left[D_{ijlm} \quad \begin{matrix} l=1\to n \\ m=1\to n \end{matrix}\right]\cdot\bar{\theta} \tag{13}$$

and $$D_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n} Trace\left[A_{kij}\cdot {}^{k}I_{psk}\cdot A_{klm}^{T}\right] \tag{14}$$

"Equation (7) and (8) is the mass matrix or inertia matrix of acceleration energies", refer to "(9)" with "(10)" and "(11)" it represents the column matrix of centrifugal and Coriolis terms, while refer to "(12)," with "(13)" and "(14)" it is named pseudo-inertial matrix of the acceleration energy. All three matrices are known in the scientific literature, for example [1], [6]-[7]. Their components illustrate on the one hand the mass proprieties included in the pseudo-inertial tensor, as:

$${}^{k}I_{psk}=\begin{bmatrix}\int {}^{k}\bar{r}_k\cdot {}^{k}\bar{r}_k^{T}\cdot dm & \int {}^{k}\bar{r}_k\cdot dm \\ \int {}^{k}\bar{r}_k^{T}\cdot dm & \int dm\end{bmatrix}= \\ =\begin{bmatrix}{}^{k}I_{pk} & M_k\cdot {}^{k}\bar{r}_{C_k} \\ M_k\cdot {}^{k}\bar{r}_{C_k}^{T} & M_k\end{bmatrix}. \tag{15}$$

On the other hand, the dynamics matrices include the so called differential matrices of first and second order, that correspond to the homogeneous transformations matrices between the reference systems of the MBS, "Fig. 1". So, the differential matrix of first order ($\bar{p}$ -position, R -orientation) shown as:

$$A_{ki(j)}=\begin{bmatrix}A_{ki(j)}(R) & A_{ki(j)}(\bar{p}) \\ 0 \quad 0 \quad 0 & 0\end{bmatrix} \tag{16}$$

$$\begin{aligned} where \quad & A_{ij}(R)=\frac{\partial}{\partial q_j}\left\{{}_{i}^{0}[R]\right\}= \\ &=\left\{\exp\left\{\sum_{k=0}^{j-1}\left(\bar{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right\}\right\}\cdot\left(\bar{k}_j^{(0)}\times\right)\cdot\Delta_j\cdot A_{ij}^{*}(R);\end{aligned} \tag{17}$$

$$A_{ij}^{*}(R)=\exp\left\{\sum_{l=j}^{i}\left(\bar{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\cdot R_{i0}^{(0)} \;; \tag{18}$$

$$\begin{aligned} and \quad A_{ij}(\bar{p})=&\left\{\exp\left[\sum_{k=0}^{j-1}\left(\bar{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right]\right\}\cdot X_j+ \\ &+\exp\left[\sum_{l=j}^{i}\left(\bar{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right]\cdot\bar{p}_i^{(0)}+A_{ij}^{*}(\bar{p});\end{aligned} \tag{19}$$

$$while \quad X_j=\left(\bar{p}_j^{(0)}\times\right)\cdot\bar{k}_j^{(0)}\cdot\Delta_j+\left(1-\Delta_j\right)\cdot\bar{k}_j^{(0)}; \tag{20}$$

$$A_{ij}^{*}(\bar{p})=\Delta_j\cdot\exp\left[\sum_{k=0}^{j-1}\left(\bar{k}_k^{(0)}\times\right)\cdot q_k\cdot\Delta_k\right]\cdot A_{ij}^{**}(\bar{p}) \;; \tag{21}$$

$$A_{ij}^{**}(\bar{p})=\sum_{l=j}^{i}\left\{\exp\left[\sum_{m=j-1}^{l-1}\left(\bar{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\cdot\delta_m\right]\right\}\cdot\bar{b}_l \tag{22}$$

The differential matrix of second order is defined below as:

$$A_{ijk}(R)=\begin{bmatrix}A_{ijk}(R) & A_{ijk}(\bar{p}) \\ 0 \quad 0 \quad 0 & 0\end{bmatrix} \tag{23}$$

$$\begin{aligned} where \quad & A_{ijk}(R)=\frac{\partial^2}{\partial q_j\cdot\partial q_k}\left\{{}_{i}^{0}[R]\right\}= \\ &=\left\{\exp\left\{\sum_{l=0}^{k-1}\left(\bar{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\right\}\cdot\left(\bar{k}_k^{(0)}\times\right)\cdot\Delta_k\cdot A_{ijk}^{*}(R);\end{aligned} \tag{24}$$

$$\begin{aligned} &A_{ijk}^{*}(R)=\left\{\exp\left\{\sum_{m=k}^{j-1}\left(\bar{k}_m^{(0)}\times\right)\cdot q_m\cdot\Delta_m\right\}\right\}\cdot A_{ijk}^{**}(R)\cdot R_{i0}^{(0)}, \\ &A_{ijk}^{**}(R)=\left(\bar{k}_m^{(0)}\times\right)\cdot\Delta_m\cdot\left\{\exp\left\{\sum_{p=m}^{i}\left(\bar{k}_p^{(0)}\times\right)\cdot q_p\cdot\Delta_p\right\}\right\};\end{aligned} \tag{25}$$

$A_{ijk}(\bar{p})=\frac{\partial}{\partial q_k}\left(A_{ij}(\bar{p})\right)$, and $A_{ij}(\bar{p})$ is given by (19).

According to [3]-[6], the sub-matrices from (16) and (23) are determined using matrix exponential functions. In their expressions the following symbols are included as follows: $\bar{k}_{k(j)}^{(0)}$ is named the unit vector, in the initial configuration, of the axis corresponding to the generalized coordinate $q_{k(j)}$, while $\Delta_{k(j)}$ is equal 1 if $q_{k(j)}$ is an angular coordinate and this is 0 otherwise. The terms $\bar{p}_i^{(0)}$ and $R_{i0}^{(0)}$ represent the position vector, respectively the orientation matrix of the system $\{i\}$ in relation to $\{0\}$, in the same initial configuration.

In conclusion of this section, refer to "(4)," it can be seen that a generalization of König's theorem from analytical dynamics but this is extended on the acceleration energies of first order.

B. Acceleration Energy of Second Order

According to the research of the main author [6]-[7], the suddenly motion of MBS, the transient motion phases, as well as the mechanical systems subjected to the action of a system of external forces, with a time variation law, are characterized by linear and angular accelerations of higher order. A simple example in agreement with this statement is the simplified mechanical system shown in "Fig. 2". Therefore, in this section the acceleration energy of second order is developed in explicit and then matrix form. "Starting from (1) and (2), equation for defining the acceleration energy of second order becomes":

$$\begin{aligned} E_A^{(2)i}&=\frac{1}{2}\int\bar{v}_i^T\cdot\bar{v}_i\cdot dm=\frac{1}{2}\int Trace\left(\bar{r}_i\cdot\bar{r}_i^{\,T}\right)\cdot dm= \\ &=\frac{1}{2}\cdot Trace\left(\bar{r}_{C_i}\cdot\bar{r}_{C_i}^T\right)\int dm+ \\ &+\frac{1}{2}\cdot Trace\left[\bar{r}_{C_i}\cdot\int {}^{i}\bar{r}_i^{*T}\cdot dm\cdot {}_{i}^{0}[R]^T\right]+ \\ &+\frac{1}{2}\cdot Trace\left[{}_{i}^{0}[R]\cdot\int {}^{i}\bar{r}_i^{*T}\cdot dm\cdot\bar{r}_{C_i}^T\right]+ \\ &\frac{1}{2}\cdot Trace\left[\int {}_{i}^{0}[R]\cdot {}^{i}\bar{r}_i^{*}\cdot {}^{i}\bar{r}_i^{*T}\cdot {}_{i}^{0}[R]^T\cdot dm\right].\end{aligned} \tag{26}$$

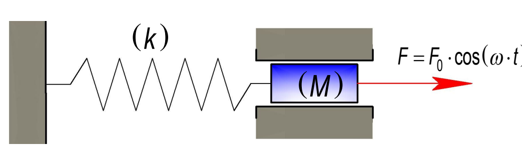


Fig. 2 Simplified mechanical system

According to [6]-[7], applying a few matrix and differential transformations, the explicit form of the acceleration energy of second order for a MBS is obtained as follows:

$$
\begin{aligned}
&E_A^{(2)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right]= \\
&=(-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot M_i\cdot{}^i\overline{v}_{C_i}^T\cdot{}^i\overline{v}_{C_i}\right\}+ \\
&+\Delta_m^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^i\overline{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\overline{\omega}_i+2\cdot{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i\times{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right)+\right. \\
&\left.+{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i\times{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right)-{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i\right\}+ \\
&+\Delta_m^2\cdot\sum_{i=1}^{n}\left\{2\cdot{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i+\right. \\
&+2\cdot{}^i\overline{\omega}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right]\cdot{}^i\overline{\omega}_i- \\
&-5\cdot\left({}^i\overline{\omega}_i^T\cdot{}^iI_{pi}^*\right)\cdot\left({}^i\overline{\omega}_i^T\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i+ \\
&+\frac{5}{2}\cdot\left({}^i\overline{\omega}_i^T\cdot{}^i\overline{\omega}_i\right)\cdot Trace\left({}^iI_{pi}^*\right)\cdot\left({}^i\overline{\omega}_i^T\cdot{}^i\overline{\omega}_i\right)+ \\
&+\frac{1}{2}\cdot{}^i\overline{\omega}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right]\cdot{}^i\overline{\omega}_i+ \\
&+{}^i\overline{\omega}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i\times{}^iI_{pi}^*\cdot{}^i\overline{\omega}_i\right)\right]\cdot{}^i\overline{\omega}_i+ \\
&\left.+\frac{1}{2}\cdot{}^i\overline{\omega}_i^T\cdot\left[{}^i\overline{\omega}_i^T\cdot\left({}^i\overline{\omega}_i^T\cdot{}^iI_i^*\cdot{}^i\overline{\omega}_i\right)\cdot{}^i\overline{\omega}_i\right]\cdot{}^i\overline{\omega}_i\right\}.
\end{aligned}
\tag{27}
$$

It can be also considered an extension of the generalization of König's theorem of the second order. Refer to "(26)" and "(27)," they include the symbols and parameters specified in the page numbers [2]. At these, the terms which are a function of $\overline{\theta}=(q_i,\ for\ \text{i}=1\to\text{n})^T$, representing the generalized accelerations of second order are also added.

According to [6], following of the application a few of matrix and differential transformations, the matrix expression of the acceleration energy of second order is determined as:

$$
\begin{aligned}
&E_A^{(2)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right]= \\
&=\frac{1}{2}\cdot\overline{\theta}^T(t)\cdot M\left[\overline{\theta}(t)\right]\cdot\overline{\theta}(t)+ \\
&+3\cdot\overline{\theta}^T(t)\cdot V\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right]+ \\
&+\overline{\theta}^T(t)\cdot H\left[\overline{\theta}(t);\overline{\theta}^2(t)\right]\cdot\overline{\theta}(t)+ \\
&+\frac{9}{2}\cdot\overline{\theta}^T(t)\cdot D\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right]\cdot\overline{\theta}(t)+ \\
&+3\cdot\overline{\theta}^T(t)\cdot K\left[\overline{\theta}(t);\overline{\theta}^4(t)\right]+ \\
&+\frac{1}{2}\cdot\overline{\theta}^T(t)\cdot N\left[\overline{\theta}(t);\overline{\theta}^4(t)\right]\cdot\overline{\theta}(t).
\end{aligned}
\tag{28}
$$

Alongside (7), the other five dynamics matrices are included in the acceleration energy of second order (27), thus:

$$V\left(\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right)=\underset{(n\times 1)}{Matrix}\left(V_i^*,\ \text{i}=1\to\text{n}\right) \tag{29}$$

where $V_i^*=\overline{\theta}^T\cdot\left[\left\{V_{ijm}=V_{imj}\right\}\quad\begin{matrix}j=1\to n\\ m=1\to n\end{matrix}\right]\cdot\overline{\theta}$ (30)

$$H\left(\overline{\theta};\overline{\theta}^2\right)=\underset{(n\times n)}{Matrix}\left[H_{ij}\quad\begin{matrix}i=1\to n\\ j=1\to n\end{matrix}\right] \tag{31}$$

$$H_{ij}=\overline{\theta}^T\cdot\left[H_{ijlm}\quad\begin{matrix}l=1\to n\\ m=1\to n\end{matrix}\right]\cdot\overline{\theta} \tag{32}$$

$$H_{ijlm}=\sum_{k=\max(i;j;l;m)}^{n}Tr\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjlm}^T\right] \tag{33}$$

$$
\begin{aligned}
&D\left(\overline{\theta};\overline{\theta};\overline{\theta}\right)= \\
&=\underset{(n\times n)}{Matrix}\left\{\overline{\theta}^T\cdot\left[D_{ijlm}\quad\begin{matrix}l=1\to n\\ m=1\to n\end{matrix}\right]\cdot\overline{\theta}\quad\begin{matrix}i=1\to n\\ j=1\to n\end{matrix}\right\}
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&K\left[\overline{\theta}(t);\overline{\theta}^4(t)\right]= \\
&=\underset{(n\times 1)}{Matrix}\left\{\overline{\theta}^T\cdot\left\{\begin{matrix}\overline{\theta}^T\cdot\left[K_{ijlmp}\quad\begin{matrix}m=1\to n\\ p=1\to n\end{matrix}\right]\cdot\overline{\theta}\\ i=1\to n;\ j=1\to n;\ l=1\to n\end{matrix}\right\}\cdot\overline{\theta}\right\},
\end{aligned}
$$

$$K_{ijlmp}=\sum_{k=\max(i;j;l;m;p)}^{n}Tr\left[A_{ki}\cdot{}^kI_{psk}\cdot A_{kjlmp}^T\right]; \tag{35}$$

$$N\left(\overline{\theta};\overline{\theta}^4\right)=\underset{(n\times n)}{Matrix}\left\{\begin{matrix}\overline{\theta}^T\cdot\left\{\begin{matrix}\overline{\theta}^T\cdot\left[N_{ijlmpr}\quad\begin{matrix}p=1\to n\\ r=1\to n\end{matrix}\right]\cdot\overline{\theta}\\ l=1\to n\quad m=1\to n\end{matrix}\right\}\cdot\overline{\theta}\\ i=1\to n\quad j=1\to n\end{matrix}\right\},$$

$$N_{ijlmpr}=\sum_{k=\max(i;j;l;m;p;r)}^{n}Tr\left[A_{kijl}\cdot{}^kI_{psk}\cdot A_{kmpr}^T\right]. \tag{36}$$

"Equations (29), (31), (34), (35) and (36) are differential matrices of second order with mass and inertia properties". The differential matrix of third order, component of the dynamics matrix (31), with (32) and (33) has the form:

$$A_{ijkm}=\begin{bmatrix} & A_{ijkm}(R) & & A_{ijkm}(\overline{p})\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{37}$$

According to [3]-[6], the sub-matrices included in (37) are expressed by means of the matrix exponential functions:

$$A_{ijkm}(R)=\left\{exp\left\{\sum_{l=0}^{m-1}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\right\}\cdot A_{ijkm}^*(R), \tag{38}$$

$$A_{ijkm}^*(R)=\left(\overline{k}_m^{(0)}\times\right)\cdot\Delta_m\cdot A_{ijkm}^{**}(R),$$

$$A_{ijkm}^{**}(R)=\left\{exp\left\{\sum_{p=m}^{k-1}\left(\overline{k}_p^{(0)}\times\right)\cdot q_p\cdot\Delta_p\right\}\right\}\cdot A_{ijkm}^{***}(R), \tag{39}$$

$$A_{ijkm}^{***}(R)=\left(\overline{k}_p^{(0)}\times\right)\cdot\Delta_p\cdot A_{ijkm}^{****}(R), \tag{40}$$

$$A_{ijkm}^{****}(R)=\left\{exp\left\{\sum_{r=p}^{j-1}\left(\overline{k}_r^{(0)}\times\right)\cdot q_r\cdot\Delta_r\right\}\right\}\cdot A_{ijkm}^{*****}(R), \tag{41}$$

$$A_{ijkm}^{*****}(R)=\left(\overline{k}_r^{(0)}\times\right)\cdot\Delta_r\cdot A_{ijkm}^{******}(R)\cdot R_{i0}^{(0)}, \tag{42}$$

$$A_{ijkm}^{******}(R) = exp\left\{\sum_{s=r}^{i}\left(\overline{k}_s^{(0)}\times\right)\cdot q_s\cdot\Delta_s\right\}; \tag{43}$$

and $$A_{ijkm}(\overline{p}) = \frac{\partial}{\partial q_k}\left[A_{ijk}(\overline{p})\right] = \frac{\partial^2}{\partial q_k\cdot\partial q_m}\left[A_{ij}(\overline{p})\right], \tag{44}$$

where the column matrix $A_{ij}(\overline{p})$ is given by equation (18).

The differential matrix of fourth order, component of the dynamics matrices (35) and (36) has the following form:

$$A_{ijkmp} = \begin{bmatrix} A_{ijkmp}(R) & A_{ijkmp}(\overline{p}) \\ 0 \quad 0 \quad 0 & 0 \end{bmatrix}; \tag{45}$$

$$A_{ijkmp}(R) = \frac{\partial^4}{\partial q_j\cdot\partial q_k\cdot\partial q_m\cdot\partial q_p}\left\{{}_i^0[R]\right\} = \tag{46}$$
$$= \left\{\exp\left\{\sum_{l=0}^{p-1}\left(\overline{k}_l^{(0)}\times\right)\cdot q_l\cdot\Delta_l\right\}\right\}\cdot A_{ijkmp}^{*}(R),$$

$$A_{ijkmp}^{*}(R) =$$
$$= \left(\overline{k}_p^{(0)}\times\right)\cdot\Delta_p\cdot\left\{\exp\left\{\sum_{r=p}^{m-1}\left(\overline{k}_p^{(0)}\times\right)\cdot q_r\cdot\Delta_r\right\}\right\}\cdot A_{ijkmp}^{**}(R),$$
$$A_{ijkmp}^{**}(R) =$$
$$= \left(\overline{k}_r^{(0)}\times\right)\cdot\Delta_r\cdot\left\{\exp\left\{\sum_{s=r}^{k-1}\left(\overline{k}_r^{(0)}\times\right)\cdot q_s\cdot\Delta_s\right\}\right\}\cdot A_{ijkmp}^{***}(R),$$
$$A_{ijkmp}^{***}(R) =$$
$$= \left(\overline{k}_s^{(0)}\times\right)\cdot\Delta_s\cdot\left\{\exp\left\{\sum_{u=s}^{j-1}\left(\overline{k}_u^{(0)}\times\right)\cdot q_u\cdot\Delta_u\right\}\right\}\cdot A_{ijkmp}^{****}(R),$$
$$A_{ijkmp}^{****}(R) =$$
$$= \left(\overline{k}_u^{(0)}\times\right)\cdot\Delta_u\cdot\exp\left\{\sum_{v=u}^{i}\left(\overline{k}_v^{(0)}\times\right)\cdot q_v\cdot\Delta_v\right\}\cdot R_{i0}^{(0)}; \tag{47}$$

$$A_{ijkmp}(\overline{p}) = \frac{\partial A_{ijkm}(\overline{p})}{\partial q_p} = \frac{\partial^2 A_{ijkm}(\overline{p})}{\partial q_m\cdot\partial q_p} = \tag{48}$$
$$= \frac{\partial^3 A_{ij}(\overline{p})}{\partial q_k\cdot\partial q_m\cdot\partial q_p} = \frac{\partial^4\overline{p}_i}{\partial q_j\cdot\partial q_k\cdot\partial q_m\cdot\partial q_p}.$$

The components of the differential matrix of fourth order, included in (45), according to same [3]-[6], it observes that they have been also determined by the matrix exponentials.

C. *Acceleration Energy of Third Order*

Considering the introductory aspects from previous section, included in the page numbers [3], the suddenly motion of MBS, the transient motion phases, as well as the mechanical systems subjected to the action of a system of external forces with a time variation law, in which the robots are included, the dynamic study is extended on the acceleration energy of third order. As example, the same "Fig.3" can be also considered. So, "using (1) and (2), in this case, in accordance with [7], the main author proposes the equation of the acceleration energy of third order". This will be defined in both variants; explicit and matrix form. First of all, the starting equation shows as:

$$E_A^{(3)}(t) = \frac{1}{2}\sum_{i=1}^{n}\int\overline{v}_i^T\cdot\overline{v}_i\cdot dm =$$
$$= \frac{1}{2}\sum_{i=1}^{n}\int Trace\left(\overline{r}_i\cdot\overline{r}_i^{\,T}\right)\cdot dm =$$
$$= \frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left\{{}_i^0[R]\cdot\left[{}^iI_{pi}^{*} + \right.\right. \tag{49}$$
$$\left.\left.+M_i\cdot{}^i\overline{r}_{C_i}\cdot{}^i\overline{r}_{C_i}^{T}\right]\cdot{}_i^0[R]^T\right\} +$$
$$+\frac{1}{2}\cdot\sum_{i=1}^{n}Trace\left[\overline{p}_i\cdot\overline{p}_i^{\,T}\right]\cdot M_i,$$

where $\overline{r}_i$ represents the absolute acceleration of third order of the elementary mass "dm", and $\overline{p}_i$ expresses the absolute acceleration of third order of the origin $O_i\in\{i\}$, where $\{i\}$ is the reference frame, in accordance with the same "Fig.1".

According to [7], applying a few matrix and differential transformations the acceleration energy of third order for MBS is obtained, under the explicit form, in the two variants:

$$E_A^{(3)}(t) = E_A^{(3)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right] + \tag{50}$$
$$+E_A^{(3)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right];$$

First variant of the acceleration energy of third order is:

$$E_A^{(3)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right] =$$
$$= E_A^{(3)}\left[\overline{\theta}(t)\right] =$$
$$= (-1)^{\Delta_m}\cdot\frac{1-\Delta_m}{1+3\cdot\Delta_m}\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot M_i\cdot{}^i\overline{v}_{C_i}^{T}\cdot{}^i\overline{v}_{C_i}\right\} +$$
$$+\Delta_m^2\cdot\sum_{i=1}^{n}\left\{\frac{1}{2}\cdot{}^i\overline{\omega}_i^T\cdot{}^iI_i^{*}\cdot{}^i\overline{\omega}_i + 3\cdot\overline{\omega}_i^T\cdot\left(\overline{\omega}_i\times I_{pi}^{*}\cdot\overline{\omega}_i\right) + \right. \tag{51}$$
$$+3\cdot\overline{\omega}_i^T\cdot\left(\overline{\omega}_i\times I_{pi}^{*}\cdot\overline{\omega}_i\right) + 3\cdot\overline{\omega}_i^T\cdot\left(\overline{\omega}_i\times I_{pi}^{*}\cdot\overline{\omega}_i\right) +$$
$$+2\cdot\left(\overline{\omega}_i\times\overline{\omega}_i\right)^T\cdot I_{pi}^{*}\cdot\left(\overline{\omega}_i\times\overline{\omega}_i\right) -$$
$$-5\cdot\overline{\omega}_i^T\cdot\left[\overline{\omega}_i^T\cdot I_i^{*}\cdot\overline{\omega}_i\right]\cdot\overline{\omega}_i - \overline{\omega}_i^T\cdot\left[\overline{\omega}_i^T\cdot I_i^{*}\cdot\overline{\omega}_i\right]\cdot\overline{\omega}_i +$$
$$\left.+\overline{\omega}_i^T\cdot\left[\overline{\omega}_i^T\cdot I_{pi}^{*}\cdot\left(\overline{\omega}_i\times\overline{\omega}_i\right)\right]\cdot\overline{\omega}_i\right\};$$

The second variant highlights the generalized variables, thus:

$$E_A^{(3)}\left[\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t);\overline{\theta}(t)\right] =$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}M_{ij}\cdot q_i\cdot q_j +$$
$$+4\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}V_{ijm}\cdot q_j\cdot q_k\cdot q_m +$$
$$+3\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}V_{ijm}^{*}\cdot q_j\cdot q_k\cdot q_m + \tag{52}$$
$$+6\cdot\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}H_{ijlm}\,q_j\cdot q_k\cdot q_l\cdot q_m +$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{p=1}^{n}K_{ijlmp}\cdot q_j\cdot q_k\cdot q_l\cdot q_m\cdot q_p.$$

"Equation (50) contains the two terms corresponding to complete form". Refer to "(51)" and "(52)," they include the symbols and parameters specified in the first two sections of this chapter. At these, only the terms which are a function of $\bar{\theta} = (q_i,\ for\ \text{i=1} \rightarrow \text{n})^T$, representing the generalized accelerations of third order are added. Similarly with the first two types of acceleration energies, it can also observe an extension of the generalization of König's theorem of the third order regarding the acceleration energy of third order.

According to [7], following of the application a few of matrix and differential transformations, the matrix expression of the acceleration energy of third order is determined as:

$$
\begin{aligned}
&E_A^{(3)}\left[\bar{\theta}(t);\bar{\theta}(t);\bar{\theta}(t);\bar{\theta}(t);\bar{\theta}(t)\right] = \\
&+\frac{1}{2}\cdot\bar{\theta}^T(t)\cdot M\left[\bar{\theta}(t)\right]\cdot\bar{\theta}(t)+ \\
&+4\cdot\bar{\theta}^T(t)\cdot V\left[\bar{\theta}(t);\bar{\theta}(t);\bar{\theta}(t)\right]+ \\
&+3\cdot\bar{\theta}^T(t)\cdot V^*\left[\bar{\theta}(t);\bar{\theta}^2(t)\right]+ \\
&+6\cdot\bar{\theta}^T(t)\cdot H^*\left[\bar{\theta}(t);\bar{\theta}^2(t)\right]\cdot\bar{\theta}(t)+ \\
&+\bar{\theta}^T(t)\cdot K^*\left[\bar{\theta}(t);\bar{\theta}^4(t)\right].
\end{aligned}
\tag{53}
$$

The dynamics matrices of third order are the following:

$$
V\left[\bar{\theta}(t);\bar{\theta}(t);\bar{\theta}(t)\right] = \underset{(n\times 1)}{Matrix}\left\{\begin{matrix}\bar{\theta}^T\cdot\left[V_{ijm}\quad\begin{matrix} j=1\rightarrow n \\ m=1\rightarrow n\end{matrix}\right]\cdot\bar{\theta} \\ where\ \ i=1\rightarrow n\end{matrix}\right\}^T; \tag{54}
$$

$$
V^*\left[\bar{\theta}(t);\bar{\theta}^2(t)\right] = \underset{(n\times 1)}{Matrix}\left\{\begin{matrix}\bar{\theta}^T\cdot\left[V_{ijm}\quad\begin{matrix} j=1\rightarrow n \\ m=1\rightarrow n\end{matrix}\right]\cdot\bar{\theta} \\ where\ \ i=1\rightarrow n\end{matrix}\right\}^T; \tag{55}
$$

$$
H^*\left[\bar{\theta}(t);\bar{\theta}^2(t)\right] = \underset{(n\times n)}{Matrix}\left\{\begin{matrix}\bar{\theta}^T\cdot\left[H_{ijlm}\quad\begin{matrix} l=1\rightarrow n \\ m=1\rightarrow n\end{matrix}\right]\cdot\bar{\theta} \\ i=1\rightarrow n;\ \ j=1\rightarrow n\end{matrix}\right\};
$$

$$
K^*\left[\bar{\theta}(t);\bar{\theta}^4(t)\right] = \underset{(n\times 1)}{Matrix}\left\{\bar{\theta}^T\cdot\left\{\begin{matrix}\bar{\theta}^T\cdot\left[K_{ijlmp}\quad\begin{matrix} m=1\rightarrow n \\ p=1\rightarrow n\end{matrix}\right]\cdot\bar{\theta} \\ i=1\rightarrow n;\ j=1\rightarrow n;\ l=1\rightarrow n\end{matrix}\right\}\cdot\bar{\theta}\right\}.
$$

The last two have the same matrix form with (33) and (35). All components are also determined with matrix exponentials described in the previous section, page numbers [4] and [5].

III. Experimental Analysis

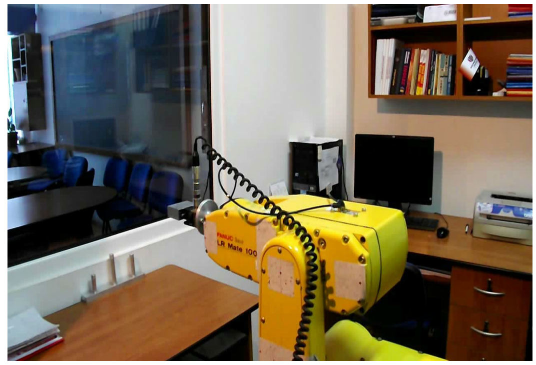


Fig. 3 Fanuc LR Mate 100iB Robot
The transducer location of the robot arm

In order to illustrate in an experimental form the validity of the above presented expressions, regarding the acceleration energies of higher order, it was considered the rotation motion on the angular interval $(0,\ \pi)$ of the arm of the serial robot Fanuc LR Mate 100 iB, according to image from "Fig. 3".

For highlight the time variation law of the acceleration energies of higher order, the polynomial interpolating functions of fifth order have been applied in the formulations from [7]:

$$
q_{ji}(\tau) = \frac{\tau_i-\tau}{t_i}\cdot q_{ji}(\tau_{i-1}) + \frac{\tau-\tau_{i-1}}{t_i}\cdot q_{ji}(\tau_i)\ ; \tag{56}
$$

$$
q_{ji}(\tau) = -\frac{(\tau_i-\tau)^2}{2\cdot t_i}\cdot q_{ji-1} + \frac{(\tau-\tau_{i-1})^2}{2\cdot t_i}\cdot q_{ji} + a_{ji1}; \tag{57}
$$

$$
\begin{aligned}
q_{ji}(\tau) &= \frac{(\tau_i-\tau)^3}{6\cdot t_i}\cdot q_{ji-1} + \\
&+\frac{(\tau-\tau_{i-1})^3}{6\cdot t_i}\cdot q_{ji} + a_{ji1}\cdot\tau + a_{ji2};
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
q_{ji}(\tau) &= -\frac{(\tau_i-\tau)^4}{24\cdot t_i}\cdot q_{ji-1} + \frac{(\tau-\tau_{i-1})^4}{24\cdot t_i}\cdot q_{ji} + \\
&+ a_{ji1}\cdot\frac{\tau^2}{2} + a_{ji2}\cdot\tau + a_{ji3};
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
q_{ji}(\tau) &= \frac{(\tau_i-\tau)^5}{120\cdot t_i}\cdot q_{ji-1} + \frac{(\tau-\tau_{i-1})^5}{120\cdot t_i}\cdot q_{ji} + \\
&+a_{ji1}\cdot\frac{\tau^3}{6} + a_{ji2}\cdot\frac{\tau^2}{2} + a_{ji3}\cdot\tau + a_{ji4}\ ;
\end{aligned}
\tag{60}
$$

where a_{jip}, $p=1\rightarrow 4$, are the integration constants, which are determined from the geometrical and kinematical constraints with an important role in ensuring the continuity of the rotation motion on the angular interval $(0,\ \pi)$, characterized by 51 interpolation segments. The obtained results, regarding the higher order polynomial functions have been included in the expressions of the acceleration energies that characterize the rotation motion of the robot arm.

By using a mono-axial accelerometer, it has been experimentally established the time variation law of the tangential component of the acceleration of a point belonging to the robot arm. Considering the rotation motion of the robot arm (the third kinetic ensemble of the robot), it results the time variation law for the angular acceleration $q_3(\tau)$. The both graphics are also represented in the "Fig. 4".

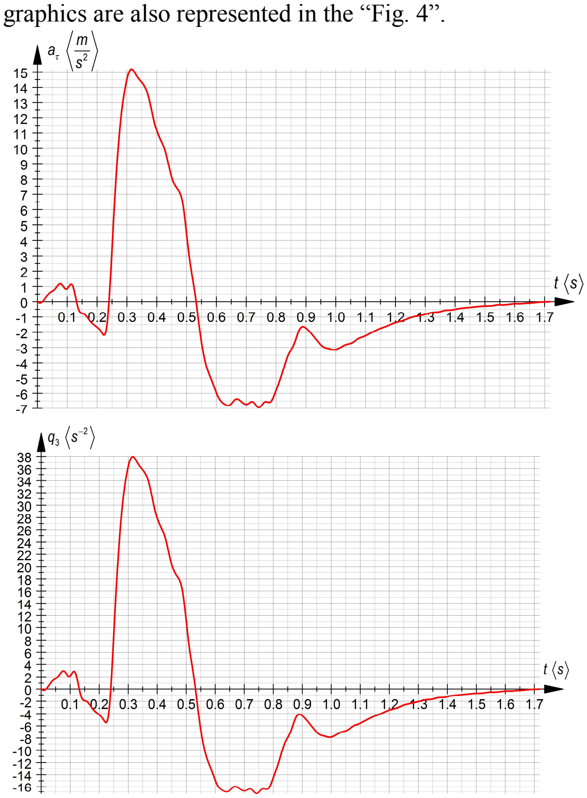


Fig. 4 The time variation of the tangential and generalized accelerations

Thus, by customizing the expressions (4), (27) and (48), for the considered application, the following expressions are:

$$E_{Aik}^{(1)}(\tau)=\frac{1}{2}\cdot\left(M_3\cdot x_{C3}^2+M_3\cdot z_{C3}^2+{}^3I_y\right)\cdot\left(q_{3ik}^2(\tau)+q_{3ik}^4(\tau)\right)$$

$$E_{Aik}^{(2)}(\tau)=\frac{1}{2}\cdot\left(M_3\cdot x_{C3}^2+M_3\cdot z_{C3}^2+{}^3I_y\right)\cdot\left[q_{3ik}^2(\tau)-\right.$$
$$\left.-2\cdot q_{3ik}^3(\tau)\cdot q_{3ik}(\tau)+9\cdot q_{3ik}^2(\tau)\cdot q_{3ik}^2(\tau)+q_{3ik}^6(\tau)\right]$$

$$E_{Aik}^{(3)}(\tau)=\frac{1}{2}\cdot\left\{\left(M_3\cdot x_{C3}^2+M_3\cdot z_{C3}^2+{}^3I_y\right)\cdot\left[q_{3ik}^8(\tau)-\right.\right. \qquad (61)$$
$$-8\cdot q_{3ik}^5(\tau)\cdot q_{3ik(\tau)}+30\cdot q_{3ik}^4(\tau)\cdot q_{3ik}^2(\tau)-$$
$$-12\cdot q_{3ik}^2(\tau)\cdot q_{3ik}(\tau)\cdot q_{3ik}(\tau)+30\cdot q_{3ik}^4(\tau)\cdot q_{3ik}^2(\tau)-$$
$$-12\cdot q_{3ik}^2(\tau)\cdot q_{3ik}(\tau)\cdot q_{3ik}(\tau)+16\cdot q_{3ik}^2(\tau)\cdot q_{3ik}^2(\tau)+$$
$$\left.\left.+24\cdot q_{3ik}(\tau)\cdot q_{3ik}^2(\tau)\cdot q_{3ik}(\tau)+9\cdot q_{3ik}^4(\tau)+q_{3ik}^2(\tau)\right]\right\}$$

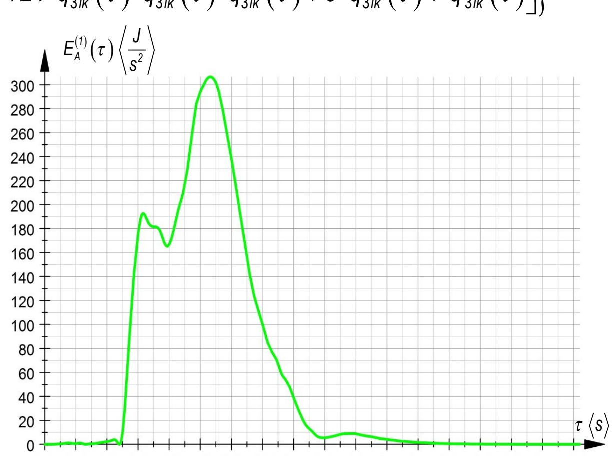


Fig. 5 Time variation law of the acceleration energy of first order

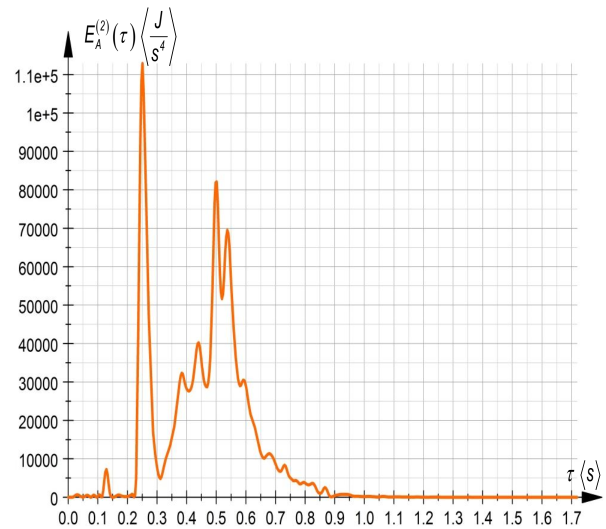


Fig. 6 Time variation law of the acceleration energy of second order

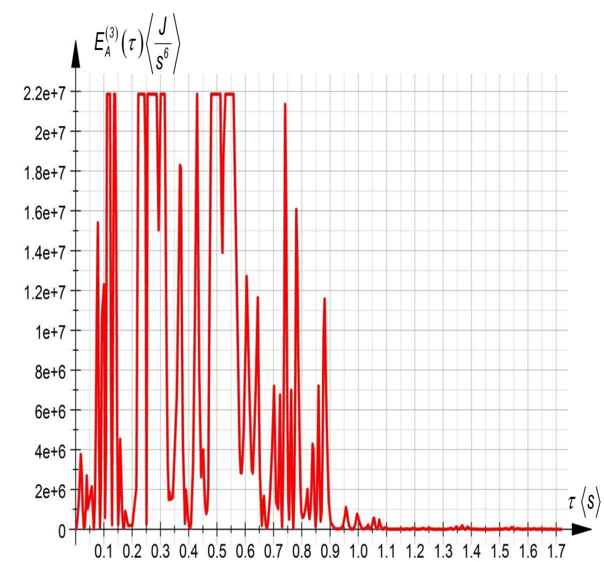


Fig. 7 Time variation law of the acceleration energy of third order

The graphics from "Fig.5", "Fig.6" and "Fig.7" illustrate the time variation law for the acceleration energies of first, second and third order in the case of the application on the serial Fanuc robot presented in the image from "Fig.3".

IV. CONCLUSION

This paper has been devoted to the presentation of important formulations on the advanced notions that are used in the analytical dynamic study of the mechanical systems. In the advanced notions the acceleration energies of higher order, also named "kinetic energies of the accelerations of higher order" are included. The existence of these energies of higher order, according to scientific literature is due to the fact that when a component of the multibody systems (MBS) is characterized by suddenly movements, there are higher order time variations of the linear and angular accelerations. This conclusion is also valid for the transient motion phase of any mechanical system. Considering the aspects from scientific literature, as well as the researches of the main author, the expressions for the acceleration energies of first, second and third orders have been presented in general, explicit and matrix form. These equations have been developed by means of the matrix exponentials functions that they have undeniable advantages in the matrix study of the kinematics and dynamics for any complex mechanical system (MBS).

The expressions of the acceleration energies can be also considered an extension of König's theorem on higher order whereas they are characterized by a resultant translation component and resultant rotation component.

Integral part of the multibody systems are the mechanical structures of the serial robots, on which an application has been presented in order to highlight the importance of the higher order energies, regarding the dynamical behavior.

REFERENCES

[1] Ardema, M., D., "Analytical Dynamics Theory and Applications", Springer US, ISBN 978-0-306-48681-4, pp. 225-243, 245-259, (2006).

[2] B.N. Frandlin, L.D. Roshchupkin, "On the Dolapchiev – Manzheron – Tsenov equations in the Case of Natural Systems", Soviet Applied Mechanics, Vol. 9, Issue 3, pp.251-254, (1973).

[3] I. Negrean, D. C. Negrean, , **"**Matrix exponentials to robot kinematics", 17th International Conference on CAD/CAM, Robotics and Factories of the Future, Vol.2, pp. 1250-1257, Durban, South Africa, (2001).

[4] Negrean, I., Duca, A.V., Negrean, D.C., Kacso, K., *"New Formulations on Acceleration Energy in the Robot Dynamics"* Proceedings of SYROM 2009, The 10th IFToMM International Symposium on Science of Mechanisms and Machines, ISBN: 978-90-481-3521-9 e-ISBN: 978-90-481-3522-6, DOI 10.1007/978-90-481-3522-6, © Springer Science+Business Media, B.V. 2009

[5] Negrean, I., Schonstein, C., Kacso, K., Duca, A., "*Matrix Exponentials and Differential Principles in the Dynamics of Robots",* The 13-th World Congress in Mechanism and Machine Science, Guanajuato, Mexico, 19-25 June, 2011. available at . http://somim.org.mx/conference_proceedings/pdfs/A12/A12_474.pdf

[6] Negrean, I., Kacso, K., Rusu, F., "*Energies of Higher Order in Advanced Dynamics of Mechanical Systems",* 2014 International Conference On Production Research - Regional Conference Africa, Europe And The Middle East And 3rd International Conference On Quality And Innovation In Engineering And Management (ICPR-AEM 2014), ISBN:978-973-662-978-5, pp. 346 – 351

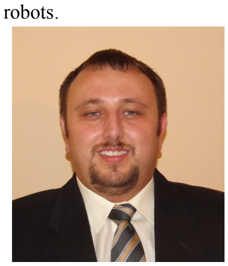
[7] Negrean, I., Kacso, K., Schonstein, C., Duca, A., "*Energies of Acceleration in Advanced Robotics Dynamics",* Applied Mechanics and Materials, ISSN: 1662-7482, vol 762 (2015), pp 67-73 Submitted: 2014-08-05 ©(2015) TransTech Publications Switzerland Revised:2014-11-16, DOI:10.4028/www.scientific.net/AMM.762.67

[8] Park, F.C., "Computational Aspects of the Product-of-Exponentials Formula for Robot Kinematics", IEEE Transaction on Automatic Control, (1994).

[9] L.A. Pars, A "Treatise on Analytical Dynamics", Heinemann, London, (2007), Vol I, pp. 1-122.

[10] F.P.J. Rimrott, B. Tabarrok, "Complementary Formulation of the Appell Equation", Technische Mechanik, Band 16, Heft 2 pp. 187-196, (1996).

Iuliu Negrean, Full Professor, PhD; (c) Member – The Academy of Technical Sciences of Romania, Technical Mechanics Section; Director of the Department of Mechanical Systems Engineering, Faculty of Machine Building, Technical University of Cluj-Napoca, Romania (2004 – 2015); Scientific PhD. Advisor in the field of Mechanical Engineering (2001); Full Professor (1999); PhD degree, in Industrial Robots field (1995); Mechanical Engineer degree, Polytechnic Institute of Cluj-Napoca, Faculty of Mechanics, Graduated as valedictorian (1980). Teaching activities: Theoretical Mechanics (1990-present); Kinematics and Dynamics of Industrial Robots (1993 – 2007); Mechanics of Robots (2008 – present); Applied Mechanics (1991 – 1992 , 2010 – 2011); Mechanics of Continuous Elastic Medium (2011 – 2013); Advanced Mechanics in Robotics (2010 – 2011); Elements of Advanced Mechanics (2005 – 2010) Calibration and Accuracy in Robotics (2010 – 2013); Trajectory Planning of robots; Reliability of Mechanical Systems (1999 – 2002, 2012). 140 scientific papers published in national and international journals and conferences, among which to 104 as main author; 15 books and monographs published (three of them in English) in publishers from Romania; Bibliographic citations in the books and scientific papers of important authors from our country and from abroad as well as in a large number of PhD. thesis; Reviewer of different academic courses and monographs; Research domains: Applied Mechanics; Robotics, Mechanical Engineering; Mathematical modeling, algorithms and simulation software regarding kinematic, dynamic and accuracy behavior of rigid and elastic mechanical structures of robots; The main contributions in the fields of research approached especially in Robotics (Mathematical modeling, algorithms and simulation software regarding kinematic, dynamic and accuracy behavior of mechanical structures of robots) can be divided into two groups of scientific results. The first group refers to new formulations and contributions brought in the field of Applied Mechanics in Robotics and respectively in Advanced Mechanics of multibody systems, among of these a few are mentioned: The algorithm of matrix exponentials in forward kinematics of robots, formulations and a series of contributions regarding the acceleration energy of the first, second and third-order in explicit and matrix form for multibody mechanical systems; formulations and contributions on differential principles based on acceleration energy of higher order; establishing of higher order differential equations, in generalized form, concerning the precise modeling for suddenly movements, respectively the transient motions of mechanical multibody systems. The second group of scientific results refers to the contributions in the field of mathematical modeling and simulation of kinematic and dynamic accuracy, in the case of the mechanical structure of the serial robots.



Kalman Kacso, Lecturer (2012) in Department of Mechanical Systems Engineering, Faculty of Machine Building, Technical University of Cluj-Napoca, Romania; PhD degree, in Mechanical Engineering field (2011); Professor Assistant in Department of Mechanical Systems Engineering, Faculty of Machine Building, Technical University of Cluj-Napoca (2007-2012). Industrial Robots degree, Technical University of Cluj-Napoca, Faculty of Machine Building (2005). Teaching activities: Theoretical Mechanics (2011-present); Mechanics of Robots (2012 – 2015). 45 scientific papers published in national and international journals and conferences, among which to 4 as main author; coauthor of 5 books and monographs published (one of them in English) in publishers from Romania. Research domains: Applied Mechanics; Robotics, Mechanical Engineering; Mathematical modeling. Contributions in Robotics are mathematical modeling of more serial structures and mobile robots; mathematical modeling of the work process using polynomial interpolation functions. The main contributions in determining the acceleration energy of the first, second and third-order; and higher order differential equations for more serial robots and graphical representation of these using programs developed in Mupad. Realization of a command program for a mobile robot and more programs for simulation of geometric command for serial and mobile structures. This program of command and simulations has developed using Visual Basic.