Gaussian Pythagorean Triples

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***Abstract* – Gaussian integer solutions of the Pythagorean equation are obtained.**

***Index Terms* – Gaussian integer, Diophantine equation, integral solutions, Pythagorean triples**

I. INTRODUCTION

The most cited Diophantine equation since antiquity till today is the Pythagorean equation represented by $X^2 + Y^2 = Z^2$. In fact, the Pythagorean equation is a treasure house containing many hidden properties. For an extensive review of various literatures, one many refer [1, 2, 3, 4, 5, 6]. In these entire above references Pythagorean triples with integer solutions has been found. The main thrust of this communication is to obtain Pythagorean triples with Gaussian integer solutions resulting in Gaussian Pythagorean triples.

II. METHOD OF ANALYSIS

The equation to be solved is

$$X^2 + Y^2 = Z^2 \qquad (1)$$

The different patterns of solutions to (1) are presented below:

A. Pattern (1):

The substitution

$$X = m + in,\ Y = (m + h) - in,\ Z = h - in \qquad (2)$$

in (1), leads to

$$(2m + h)^2 = 2n^2 + h^2 \qquad (3)$$

Case (I)

Solving (3) as a Pellian, we get

$$m_k = \frac{h}{4}\alpha_k^2 ,$$
$$n_k = \frac{h}{2\sqrt{2}}\alpha_k \beta_k .$$

where

$$\alpha_k = (\sqrt{2}+1)^{k+1} - (\sqrt{2}-1)^{k+1}$$
$$\beta_k = (\sqrt{2}+1)^{k+1} + (\sqrt{2}-1)^{k+1}$$

Substituting these values in (2), we obtain the Gaussian integer solutions of (1) as

$$X_k = \frac{h}{4}\alpha_k^2 + i\frac{h}{2\sqrt{2}}\alpha_k\beta_k$$
$$Y_k = \frac{h}{4}\beta_k^2 + i\frac{h}{2\sqrt{2}}\alpha_k\beta_k$$
$$Z_k = h - i\frac{h}{2\sqrt{2}}\alpha_k\beta_k$$

The solutions satisfy the following recurrence relations.

1. X_{k+2}- $6X_{k+1} + X_k = 2h$
2. Y_{k+2}- $6Y_{k+1} + Y_k = -2h$
3. Z_{k+2}- $6Z_{k+1} + Z_k = -4h$

Case (II)

(3) is also satisfied by

$2m + h = 2r^2 + s^2$
$n = 2rs$
$h = 2r^2 - s^2$

Using these values in (2), we get the solutions of (1) as

$$X = s^2 + i2rs$$
$$Y = 2r^2 - i2rs$$
$$Z = (2r^2 - s^2) - i2rs$$

A few relations among the solutions are as follows.

(1) $3(X + Z), 6(Y - Z)$ represents Nasty numbers.

(2) If r and s are taken as the legs of the Pythagorean triangles, then $3(X + 2Y - Z)$ is a Nasty number.

(3) When $r = 2pq, s = 2p^2 - q^2$, or $p^2 - 2q^2$, $6(X + Y)$ is a Nasty number.

(4) $2X + Y + Z$ is a perfect square when $r = ab,\quad s = a^2 - b^2, a > b$

(5) When $r = 2^{\alpha-1}, s = 1, 2X + Y + Z = 3J_{2\alpha} + 2,\ 3J_{2\alpha} + 2 = j_{2\alpha}$ where $J_n = \frac{1}{3}[2^n - (-1)^n]$ and $j_n = [2^n + (-1)^n]$ represent Jacobsthal and Jacobsthal Lucas numbers respectively.

(6) $(Y - Z - X)^2 + 2(Y - Z)(X + Z) = 0.$

Case (III)

(3) can also be simplified as

$2m + h = r^2 + 2s^2$
$n = 2rs$
$h = r^2 - 2s^2$

Using these values in (2), we get the solutions of (1) as

$$X = 2s^2 + i2rs$$
$$Y = r^2 - i2rs$$
$$Z = (r^2 - 2s^2) - i2rs$$

B. Pattern (2):

The substitution

$X = a - ib$
$Y = a + ic$
$Z = a + id$

reduces (1) to

$$a^2 = 2bc \qquad (4)$$
$$d = c - b \qquad (5)$$

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Case (IV)

Consider the linear transformations

$$2b = u + v, c = u - v$$

in (4) we get

$$u^2 = a^2 + v^2 \quad (6)$$

which can be solved as

$$u = r^2 + s^2$$
$$a = r^2 - s^2$$
$$v - 2rs$$

Thus we get

$$a = r^2 - s^2$$
$$b = \frac{u+v}{2} = \frac{r^2 + s^2 + 2rs}{2}$$
$$c = u - v = r^2 + s^2 - 2rs$$
$$d = \frac{r^2 + s^2 - 6rs}{2}$$

To obtain integer solutions of (1), put $r = 2R, s = 2S$

There fore

$$X = (4R^2 - 4S^2) - i(2R^2 + 2S^2 + 4RS)$$
$$Y = (4R^2 - 4S^2) + i(4R^2 + 4S^2 - 8RS)$$
$$Z = (4R^2 - 4S^2) + i(2R^2 + 2S^2 - 12RS)$$

Case (V)

Considering (4) and (5), we obtain the quadratic equation

$$(c + b)^2 = d^2 + 2a^2$$

which can be solved as

$$c + b = 2r^2 + s^2$$
$$u - 2rs$$
$$d = 2r^2 - s^2$$

Thus the solutions of (1) are

$$X = 2rs \quad is^2$$
$$Y = 2rs - i2r^2$$
$$Z = 2rs - i(2rs - s^2)$$

Case (VI)

The introduction to the linear transformations

$$b = u + v, c = u - v, bc = 2\alpha^2$$

in (4) gives

$$u^2 = 2\alpha^2 + v^2$$

which can be solved and thus we obtain two different patterns of Gaussian integral solutions of (1) as

$$X = 4rs - i4r^2$$
$$Y = 4rs - i2s^2$$
$$Z = 4rs - i(2s^2 - 4r^2)$$

and

$$X = 4rs - i2r^2$$
$$Y = 4rs - i4s^2$$
$$Z = 4rs - i(4s^2 - 2r^2)$$

Case (VII)

In $bc = 2\alpha^2$ taking

$$b = u + h, c = u - h \quad (h \text{ is a non zero integer}),$$

we get the Pellian

$$u^2 - 2\alpha^2 + h^2$$

Thus, after some algebra, the Gaussian integral solutions of (1) are

$$X_k = \frac{h}{\sqrt{2}}\alpha_k\beta_k - i\frac{h}{2}\alpha_k^2$$
$$Y_k = \frac{h}{\sqrt{2}}\alpha_k\beta_k + i\frac{h}{2}\beta_k^2$$
$$Z_k = \frac{h}{\sqrt{2}}\alpha_k\beta_k - i2h$$

C. Pattern (3):

The substitution

$$X = a + it$$
$$Y = b + it$$
$$Z = c + it$$

reduces (1) to

$$t^2 = -2bc \quad (7)$$
$$c = a + b \quad (8)$$

Taking $ab = -2\alpha^2$ and considering (7) and (8), we obtain the quadratic equation

$$(a - b)^2 = c^2 + 8\alpha^2$$

which can be solved and thus we obtain two different patterns of Gaussian integral solutions of (1) as

$$X = 8r^2 + i4rs$$
$$Y = -s^2 + i4rs$$
$$Z = (8r^2 - s^2) + i4rs$$

and

$$X = r^2 + i4rs$$
$$Y = -8s^2 + i4rs$$
$$Z = (r^2 - 8s^2) + i4rs$$

CONCLUSION

One many search for other patterns of solutions.

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