On Evaluations Some Sepecial Functions with Pade Approximant in Mupad Interface and Mathematica

D.A.Gismalla

Abstract— The stability convergence for each method depends on the particular method chosen for that particular function to be approximated. However, it is well known that methods for approximations are classified in ascending order according to their efficiency and stability to do more better in approximation. These techniques in that order are PADE, CONTINUED FRACTION, CHEBYESH and POWER SERIES. Despite the fact that Taylor’s series are not so worst in general, many methods are developed for series acceleration as Euler’s or Levin’s Transformations. This means that an experienced numerical analyst can easily select which method is going to choose directly without runs into many complicated difficulty. So this the idea behind this paper is to compared some methods against the other in such away that the reader can be acquainted to know which method is better than other.

Hence, this paper first give few commands in MUPAD INTERFACE and MATHEMATICA to get directly to approximate functions in a continued fraction or PADE form. second, a brief description for PADE is given with one of its important application in finding the root of equations. The Method is called Halley’s algorithm that compared against Newton’s method for finding the roots. Third, methods that we will consider are Gaussian quadrure, Pade Approximant, Continued Fraction and Levin’s Transform to be compared and tested with examples showing which method is applicable and preferable than the other when they applied to a particular problem

Example 1 Continued Fraction Approximation
In the command to express the Taylor’s expansion for exp(x) around x=0 in a continued fraction form

\[ \frac{1}{1 + \frac{x}{1 + \frac{x^2}{1 + \frac{x^3}{1 + \frac{x^4}{1 + \ldots}}}}} \]

\[ \text{>> conffrac(exp(-3*x^2), x = 0)} \]

Example 2 Series Approximation
We compute a Laurent expansion around the point x=1

\[ \text{>> s := series(1/(x^2 - 1), x = 1)} \]

Example 3 Pade approximation

\[ \text{>> ?pade} \]

Syntax
pade(f, x, [m, n])
pade(f, x = x0, [m, n])

Description
pade(f, ...) computes a Pade approximant of the expression f. The Pade approximant of order \([m, n]\) around \(x = x_0\) is a rational expression

\[ \frac{(X - x_0)^p \{ a_0 + a_1 (X - x_0) + \ldots + a_m (X - x_0)^m \}}{1 + b_1 (X - x_0) + \ldots + b_n (X - x_0)^n} \]

approaching \(f\). The parameters \(p\) and \(a_0\) are given by the leading order term \(f = a_0 (x - x_0)^p + O((x - x_0)^{p+1})\) of the series expansion of \(f\) around \(x = x_0\). The parameters \(a_1, \ldots, b_n\) are chosen such that the series expansion of the Pade approximant coincides with the series expansion of \(f\) to the maximal possible order.

The expansion points infinity, -infinity, and complexInfinity are not allowed.

If no series expansion of \(f\) can be computed, then FAIL is returned. Note that series must be able to produce a Taylor series or a Laurent series of \(f\), i.e., an expansion in terms of integer powers of \(x - x_0\) must exist.

The Pade approximant is a rational approximation of a series expansion:

\[ \text{>> f := cos(x)/(1 + x): P := pade(f, x, [2, 2])} \]

\[ \frac{-7x^2 + 2x + 12}{x^2 + 14x + 12} \]

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II. COMMANDS IN MATHEMATICA

Example 1 (power series approximation in Mathematica)
This gives a power series approximation to \((1+x)^n\) for \(x\) close to 0, up to terms of order \(x^3\).

```plaintext
In[1]:= Series[Exp[-t], {t, 0, 3}]
Out[1]= 1 + t + 1/2(-1 + n + n x^2 + 1/6 (-2 + n - 1 + n) n x^3) + O[x]^4
```

Mathematica knows the power series expansions for many mathematical functions.

```plaintext
In[2]:= Series[Exp[a t] (1 + Sin[2 t]), {t, 0, 4}]
Out[2]= 1 + (2 + a t)(-2 a + a^2)/2 t^2 + (6 + 3 a^2 - a^3)/6 t^3 + 1/2(32 a - 8 a^3 + a^4) t^4
```

Example 3 (Pade Approximant in Mathematica)

```plaintext
PadeApproximant[expr, {x, x0, {m, n}}] gives the Padé approximant to expr about the point \(x=x_0\), with numerator order \(m\) and denominator order \(n\).
PadeApproximant[expr, {x, x0, n}] gives the Padé approximant to expr about the point \(x=x_0\), of order \(n\).
```

Order [2/3] Padé approximant for \(\exp(x)\):

```plaintext
In[1]:= PadeApproximant[Exp[x], {x, 0, {2, 3}}]
Out[1]= 1 - 2 x + x^2/6
```

PadeApproximant can handle functions with poles:

```plaintext
In[2]:= PadeApproximant[Exp[x]/x, {x, 0, {2, 3}}]
Out[2]= 1 - x + x^2/20 - x^3/60
```

III. PADE TABLE FOR PADE APPROXIMANT

A function \(f(z)\) is represented by a formal power series:

\[
f(z) = c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{j=0}^{\infty} c_j z^j,
\]

where \(c_0 \neq 0\), by convention. The \((m, n)\)th entry \(R_{m,n}\) in the Padé table for \(f(z)\) is then given by

\[
R_{m,n}(z) = \frac{P_m(z)}{Q_n(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}
\]

where \(P_m(z)\) and \(Q_n(z)\) are polynomials of degrees not more than \(m\) and \(n\), respectively. The coefficients \(\{a_i\}\) and \(\{b_i\}\) can always be found by considering the expression

\[
f(z) \approx \sum_{j=0}^{m+n} c_j z^j =: f_{\text{approx}}(z)
\]

\[
Q_m(z) f_{\text{approx}}(z) = P_n(z)
\]

\[
Q_m(z); f_{\text{approx}}(z) = P_n(z)
\]

and equating coefficients of like powers of \(z\) up through \(m + n\). For the coefficients of powers \(m + 1\) to \(m + n\), the right hand side is 0 and the resulting system of linear equations contains a homogeneous system of \(n\) equations in the \(n + 1\) unknowns \(b_i\) and so admits of infinitely many solutions each of which determines a possible \(Q_n\). \(P_n\) is then easily found by equating the first \(m\) coefficients of the equation above. However, it can be shown that, due to cancellation, the generated rational functions \(R_{m,n}\) are all the same, so that the \((m, n)\)th entry in the Padé table is unique. Alternatively, we may require that \(b_0 = 1\), thus putting the table in a standard form.

Although the entries in the Padé table can always be generated by solving this system of equations, that approach is computationally expensive. More efficient methods have been devised, including the epsilon algorithm.

If the difference of \(Q(z)f(z) - P_n(z)\) having the first term with degree \(n+m+r+1\), for \(r>0\), then the rational function \(R_{m,n}\) occupies \((r + 1)\) cells in the Padé table, from position \((m, n)\) through position \((m+r, n+r)\), inclusive. The Padé Table is called normal for the function \(\exp(x)\) that can be constructed using the MUPAD COMMAND as in the following Table I and it is not
normal for $\sin(x)$-1 as in Table II. The Pade approximant has many application in Physics and Mathematics for there much connection between it and the Continued Fraction Technique. It has a wide application specially in approximations and solving a system of non-linear equation as Halley's algorithm emerged from it to find the roots of a polynomial.

<table>
<thead>
<tr>
<th>$n,m$=0(1,3)</th>
<th>1</th>
<th>0</th>
<th>$-\frac{1}{x+1}$</th>
<th>$-\frac{2}{x^2 - 2x + 2}$</th>
<th>$-\frac{6}{x^3 - 3x^2 + 6x - 6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x+1$</td>
<td>$x+1$</td>
<td>$x+2$</td>
<td>$2(x+3)$</td>
<td>$6(x+4)$</td>
<td></td>
</tr>
<tr>
<td>$\frac{x^2}{2} + x + 1$</td>
<td>$\frac{x^2}{2} + x + 1$</td>
<td>$\frac{x^2 + 4x + 6}{2(x-3)}$</td>
<td>$\frac{x^2 + 6x + 12}{x^2 - 6x + 12}$</td>
<td>$\frac{3(x^2 + 8x + 20)}{x^3 - 9x^2 + 36x - 60}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{x^3}{6} + \frac{x^2}{2} + x + 1$</td>
<td>$\frac{x^3}{6} + \frac{x^2}{2} + x + 1$</td>
<td>$\frac{x^3 + 6x^2 + 18x + 24}{6(x-4)}$</td>
<td>$\frac{x^3 + 9x^2 + 36x + 60}{3(x^2 - 8x + 20)}$</td>
<td>$\frac{x^3 + 12x^2 + 60x + 120}{x^2 - 12x^2 + 60x - 120}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n,m$=0(1,4)</th>
<th>1</th>
<th>0</th>
<th>$-\frac{1}{x+1}$</th>
<th>$-\frac{1}{x^2 + x + 1}$</th>
<th>$-\frac{6}{5x^3 + 6x^2 + 6x + 6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x-1$</td>
<td>$x-1$</td>
<td>$5x-6$</td>
<td>$\frac{6(4x-5)}{x^3 + 6x^2 + 6x + 30}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x-1$</td>
<td>$x-1$</td>
<td>$\frac{x^2 - 6x + 6}{x^2 + 6}$</td>
<td>$\frac{3(15x^2 - 34x + 20)}{7x^3 - 3x^2 + 42x - 60}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{x^3}{6} + x - 1$</td>
<td>$-\frac{x^3}{6} + x - 1$</td>
<td>$\frac{7x^3 + 3x^2 - 60x + 60}{3(x^2 + 20)}$</td>
<td>$\frac{7x^3 + 3x^2 - 60x + 60}{3(x^2 + 20)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-\frac{x^3}{6} + x - 1$</td>
<td>$-\frac{x^3}{6} + x - 1$</td>
<td>$\frac{7x^3 + 3x^2 - 60x + 60}{3(x^2 + 20)}$</td>
<td>$\frac{7x^3 + 3x^2 - 60x + 60}{3(x^2 + 20)}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**IV. SOLVING KEPLER'S EQUATION USING PADE'S APPROXIMANTS**

In order to demonstrate the application of the Pade' approximants to a problem revelant to Astronomy, consider Kepler's equations

$E-\sin(E)=M, M \in (0, 2\pi), E \in [0,1]$  

which yields

$f(x) = x - \sin(x)$  

and

$f'(x) = 1 - \cos(x)$  

$f''(x) = \sin(x)$

It has been shown in [1], pp. 24, the correction $\Delta x_1$ derived from the Pade approximant of order $(1,1)$ is

$$\Delta x_1 = -\frac{ff'}{(f')^2 - \frac{1}{2}f''}, t = 0, 1, 2, \ldots \quad (2)$$

Eqn. (1) is called Halley's algorithm for finding a root from a non-linear equation. Now, Halley's algorithm is similar to Newton's iteration technique given by

$$\Delta x_i = -\frac{ff'}{(f')^2 - \frac{1}{2}f''}, t = 0, 1, 2, \ldots \quad (2)$$

where

$$\Delta x_i \text{ forward difference operator on } x_i$$

Both the techniques can be used to solve a system of non-linear equations provided that the initial starting guess solution $x_0$ is given. Now to find the root for Kepler's Eqn.(3), We write for Halley's algorithm

$$\Delta x_i = \frac{(x - \sin(x)) - x_0}{(1 - \cos(x)) - (x - x_0)}, t = 0, 1, 2, 3 \quad (3)$$

while we write for Newton's Raphson Method Eqn.(4)

$$\Delta x_i = \frac{(x - \sin(x)) - x_0}{(1 - \cos(x))}, t = 0, 1, 2, 3 \quad (4)$$
It is well known that Newton's Method fails whenever the initial starting solution is far from the real exact solution and since Halley's algorithm has the order of convergence as Newton it may diverge similarly. Now with $M=0.6$, $e=0.9$ and $x_0=0.08$, the result solution for both method is $x=1.497589413390409$

V. EVALUATION ON SOME SPECIAL FUNCTIONS

Debye function A.

The idea for stability and efficient approximating result is achieved when a suitable method is applied to a particular function. Here, for example, we apply Gaussian rule, Pade approximant and then integrate with Gaussian rule, on Debye function given in [2], page (998) which is defined as

$$\int_0^{\infty} \frac{e^{-x}}{x^2} \, dx = n! \zeta(n+1)$$

(5)

where $n!$ means factorial $n$ while $\zeta(n+1)$ is the Zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_p (1 - p^{-s})^{-1}$$

(6)

where

In [2], the handbook of mathematical functions edited by Milton and Stegun a table evaluating the Debye function from $x=0$ to 10 and $n=1$ to 4 to evaluate the integral

$$\int_0^{\infty} \frac{e^{-x}}{x^2} \, dx$$

(7)

with no hint describing the methods they used but certainly with a computer of high accuracy for decimal places. We evaluate it for $x=0.1$ and $x=10$ while $n=1$ first and second $n=4$. The procedure that we adopted is Gaussian Quadrature with five nodes as in Fig.(1) for its Matlab program. The next approach that we applied when we get advantage of the Pade command in MUPAD interface to approximate the integrand function in Eqn. (7) by a rational pade(n,m) and then Gaussian Rule again with five nodes is applied. The result are collected from command window figures as in Fig.(2) but some values are obtained when apply the command Pade first and then we apply Gaussian Rule (( without given the command window here but cited the result only )) and then all are inserted in Debye function Table III and Table IV.
function [Ie]=quassianTable(f,a,b,n)
% Example Integration using Guassian Quatrature rules
% In Matlab command window
% sym x;
% f=inline('x^4/(exp(x)-1)');
% a=0; b=0.5;
% n=5; % Guassian-Table is given and n runs from 2 to 5
% [Ie]=quassianTable(f,a,b,n)

c=[1.0000000000 0.5555555556 0.3478548451 0.2369268850
   1.0000000000 0.8888888889 0.6521451549 0.4786286705
   0.0000000000 0.5555555556 0.6521451549 0.5688888889
   0.0000000000 0.0000000000 0.3478548451 0.4786286705
   0.0000000000 0.0000000000 0.0000000000 0.2369268850];
x=[0.5773502692 0.7745966692 0.8611363116 0.9061798459
   -0.5773502692 0.0000000000 0.3399810436 0.5384693101
   -0.7745966692 -0.3399810436 0.0000000000 0.0000000000
   -0.8611363116 -0.8611363116 -0.3399810436 -0.5384693101
   0.0000000000 0.0000000000 0.0000000000 0.0000000000 -0.9061798459];

sum=0;
for j=1:n
   t=((b-a)*x(j,n-1)+a+b)/2;
   sum=sum+c(j,n-1)*feval(f,t)*(b-a)/2;
end
Ie=4*sum/b^4:

Fig.(1) gaussianTable.m file for qadtrature

sym x;
a=0; b=10;
n=5;
f=inline('x/(exp(x)-1)');
[Ie]=gaussianTable(f,a,b,n)

Ie = 0.1644
b=0.5;
[Ie]=gaussianTable(f,a,b,n)

Ie = 0.8819

Fig.(2) Gaussian for Debye function when n=1 & 4, x=0.1 and x=10

Table III Evaluation of Debye function when n=1, x=10 and x=1

<table>
<thead>
<tr>
<th>x</th>
<th>Apply Gaussian only</th>
<th>Apply Padé and Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.8819</td>
<td>0.8819</td>
</tr>
<tr>
<td>10.0</td>
<td>0.1644</td>
<td>0.1644</td>
</tr>
</tbody>
</table>
Table IV  Evaluation of Debye function when n=4 , x=0.1  and x=10

<table>
<thead>
<tr>
<th>x</th>
<th>Apply Gaussian only</th>
<th>Apply Pade &amp; Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.8138</td>
<td>0.8138</td>
</tr>
<tr>
<td>10.0</td>
<td>0.0097</td>
<td>0.0134</td>
</tr>
</tbody>
</table>

The reader should observed that the values for integral in the second column in Table III and Table IV are obtained when n=1 with command Pade approximant as  

\[
pade(x/(\exp(x)-1), x,[4,8])
\]

15120 \left( \frac{x^4-40 x^3+660 x^2-5200 x+17280}{x^4-65 x^3+100 x^2+76 x+17640} \right) - \frac{30 x^3 (x^4-28 x^3+336 x^2-2016 x+5040)}{x^5+30 x^4+420 x^3+5040 x^2+15120 x+151200} \]

while for n=4 with command Pade approximant as  

\[
pade(x^4/(\exp(x)-1), x,[4,5])
\]

30 \left( \frac{x^4-28 x^3+336 x^2-2016 x+5040}{x^5+30 x^4+420 x^3+5040 x^2+15120 x+151200} \right)

Now, We apply Gaussian Rule with 5 points to get 0.013 which is the worst result of the four results given in Table III and Table IV. This certainly due to the accumulation of rounding errors and the precision of our evaluation is not high and Gaussian Rule can't attained higher accuracy more than 9 digits of places. Also, We observe the order in the command in Eqn.(8) is Pade[4,5] but the Pade Approximant is of order 8 only. All these will effect the approximation results. Even, here there is another plenty one must adjust the file qaussianTable.m given in Fig.(1) for the denominator and numerator are two long to be submitted as parameters easily.

Lastly, both these techniques are very important and each one is suitable for certain problems, e.g. Pade can be designed for problems having poles or singularity inside the domain or at its end points.

B. The Zeta and Eta Functions

Next, as a typical example of conditional convergent series, the eta function \( \eta(z) \)

\[
\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z} , \quad \text{Re}(z) > 0 \quad (9)
\]

Which is connected to Riemann's zeta function given by

\[
\eta(z) = (1 - 2^{1-z}) \zeta(z) , \quad z \neq 0 \quad (10)
\]

where Riemann's zeta

\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} , \quad \text{Re}(z) > 1 \quad (11)
\]

or, alternatively by

\[
(1 - 2^{1-z}) \zeta(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^z} , \quad \text{Re}(z) > 0 , z = -1 \quad (10)
\]

A series which converge for \( \sigma > 0 \), i.e. also in the strip \( 0 < \sigma < 1 \).

By use of more advanced methods the following relation can be hold

\[
\zeta(z) = 2^{1-z} \pi^{-z} \sin \left( \frac{\pi z}{2} \right) \Gamma(1-z) \zeta(1-z) \quad (11)
\]

through this relation  Eqn.(11) , the function can be computed when \( \sigma > 0 \). Alternatively useful representation is given by

\[
\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}}{e^x - 1} \text{dx} , \quad \sigma > 1 \quad (12)
\]

The formula in Eqn.(12) can be transformed to

\[
\zeta(z) = \frac{e^{-x z}}{2\pi i} \int_C \frac{e^{\zeta z}}{z^{z-1}} \text{dz} \quad (13)
\]

where C is the real axis from \(-\infty \) to \( \infty \), the circle \( |z| = \pi \), and again the real axis from \( \sigma \) to \( \infty \). Eqn.(13) implies that when \( s=0 \) can be written as

\[
\zeta(0) = \frac{1}{2\pi i} \int_C \left( e^z/2 \right) \text{dz} = -\frac{1}{2} \quad (14)
\]
Alternatively, instead of using contour integration, the divergent series for the Eta function

\[ \eta(\infty) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \eta(0) = 1 - 1 + 1 - \cdots (1b) \]

Hence, using suitable manipulating we find that \( \eta(0) = -\frac{\pi^2}{6} \rightarrow \zeta(0) = -\frac{\pi^2}{6} \)

A very nice discussion can be found in [3] from which the zeros and the calculated values for \( \zeta(\pm s) \) as, We cited some from page 113 to be

\[ \zeta(-2n) = 0, \]
\[ \zeta(1 - 2n) = \frac{(-1)^{n+1}}{2n} \]
\[ \zeta(-2m) = \frac{1}{2m} \frac{(2m)!}{(2m-1)!} \]

where \( B_m \), \( m = 1, 2, 3 \), and \( -2, -4, -6 \ldots \) are trivial zeros of zeta function.

From [3], we cited some numerical values for the zeta functions as in Table V exactly as they are and then we evaluate these values again to be compared with our results. We apply Levin's Transform in [6] with its program MATLAB as in Fig.(4.7), pp27. In fact we evaluate the Eta function given by Eqn.(9) and simultaneously from the result of the program the Zeta function is evaluated by Eqn.(10). Table VI shows the values of Eta and Zeta where the number of terms is 12 and the number of decimal places for accuracy is at least 10. Observes that when \( s = -1 \)

Levin's Transform converge 0.25 which is the value for Eta function when \( s = -1 \) which gives the value for Zeta function

\[ \zeta(s) \approx 1/12 \] and \( \zeta(0) \approx -1/2 \)

**Table V** Zeta values we cited from [3]

<table>
<thead>
<tr>
<th>S</th>
<th>( \zeta(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/2</td>
<td>1.3414872573</td>
</tr>
<tr>
<td>3/2</td>
<td>2.6123753487</td>
</tr>
<tr>
<td>4/3</td>
<td>3.6009377506</td>
</tr>
<tr>
<td>1/2</td>
<td>-1.4603545088</td>
</tr>
<tr>
<td>1/3</td>
<td>-0.9733602458</td>
</tr>
<tr>
<td>0</td>
<td>-0.5</td>
</tr>
<tr>
<td>-1/2</td>
<td>-0.2078862250</td>
</tr>
<tr>
<td>-1</td>
<td>-0.0833333333</td>
</tr>
</tbody>
</table>

**Table VI** Values we computed using Levin for Eta & Zeta

<table>
<thead>
<tr>
<th>n</th>
<th>s</th>
<th>( \zeta(s) )</th>
<th>( \eta(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>5/2</td>
<td>1.341487257250917</td>
<td>0.867199889012184</td>
</tr>
<tr>
<td>12</td>
<td>3/2</td>
<td>2.612375348685454</td>
<td>0.765147024625398</td>
</tr>
<tr>
<td>12</td>
<td>4/3</td>
<td>3.600937750458803</td>
<td>0.742871563883650</td>
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<tr>
<td>12</td>
<td>1/2</td>
<td>-1.460354508809559</td>
<td>0.604898643421619</td>
</tr>
<tr>
<td>12</td>
<td>1/3</td>
<td>-0.973360248350768</td>
<td>0.571752833825269</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td>12</td>
<td>-1/2</td>
<td>-0.207886224977360</td>
<td>0.380104812609694</td>
</tr>
<tr>
<td>-1</td>
<td>-1/12</td>
<td></td>
<td>0.25</td>
</tr>
</tbody>
</table>
function LevinTranfSumx(f,n,a0,x)

% Levin Transform for Series Summation with each term having variable x
% This Technique is a series Technique for which the integral is expressed
% as a series FIRST and then LEVIN is applied. The general term for the
% series can be expressed as an inline function or a handle object with
% the initial term submitted to the program in advanced to generate the
% other terms with the number of terms n to be taken for the sum.
% The function f in LevinTranfSum(f,n,a0,x) is a ratio to generate other terms.
% In Matlab command window
% sym s;
% f=inline('(-1)^s*(2*s-1)/(2*s*(2*s+1))');
% a0=x;
% n=5;
% LevinTranfSum(f,n,a0,x)
global UT;
for k=1:n
    [S,UT]=LevinTranfSumx(f,k,a0,x);
    F(k)=UT;
end
disp('The Sum of the Series having 2*n+2 terms')
disp([ S ]); disp(F); disp('The Sum of the Series using LEVIN TRANSFORM using 2*n+2 terms')
disp([ F ]); function [ S , UT]=LevinTranfSumx(f, k, a0 , x)
a(1)=a0;
S(1)=a(1);
C(1)=1;
TotalSumDen(1)=1;
TotalSumNum(1)=1;
for j=1:2*k+1
    a(j+1)=feval(f,j)*x^2*a(j);
    S(j+1)=S(j)+a(j+1);
    C(j+1)=(2*k+1-j)*C(j)/j;
    TotalSumDen(j+1)=TotalSumDen(j)+(-1)^j*C(j+1)/a(j+1);
    TotalSumNum(j+1)=TotalSumNum(j)+(-1)^j*C(j+1)/(2*k-1)/a(j+1);
end
UT=TotalSumDen(2*k+2)/TotalSumNum(2*k+2);

Fig.(3) File LevinTranfSum.m for series with terms having variable x

VI. TRANSFORM INTEGRALS TO CONTIUED FRACTION

If, we consider the integral
\[ I(m, n) = \int_{0}^{1} \frac{x^{m-1}}{1 + x^n} \, dx \quad (17) \]

and after integrating with x=1, the value s will be
\[ s = \frac{1}{m} - \frac{1}{m + n} + \frac{1}{m + 2n} - \frac{1}{m + 3n} + \ldots \quad (18) \]

It can be shown from theorem (2.1) in [5] the Continued Fraction is
\[ \frac{1}{s} = m + \frac{n^2}{(m + n)^2 + \frac{n}{m + 2n} + \frac{n}{m + 6n} + \ldots} \quad (19) \]

Further, if we again consider the integral in Eqn.(17) when m=1 and n=2, its value actually \( \frac{1}{s} \) when x=1 and the integral for general x is of the form in Eqn.(20)
\[ I(2,1) = \int_0^1 \frac{x^{2n-1}}{(2n-1)x^2} \, dx \quad (20) \]

and it can be represented as C.F. in Eqn. (21) such as
\[ I(2,1) = \frac{1}{(2n-1)} \int_{0}^{1} x^{2n-1} \, dx \]
\[ = \frac{1}{(2n-1)} \cdot \frac{1}{2n} \quad (21) \]

Now if we put \( n = 1 \), we get as in \([5]\), the C.F. for \( \frac{\pi}{4} \) to be
\[ \frac{\pi}{4} = \frac{1}{2} \frac{1}{1+2^2} \frac{1}{1+2^2} \ldots \quad (22) \]

Hence, the computed value of the integral in Eqn. (17) for \( m=n=1 \) is in Table VII having the exact value \( I(1,1)=\log(2) \) while the computed value of the integral in Eqn. (17) for \( m=1 \) and \( n=2 \) is in Table VII having the exact value \( I(2,1)=\frac{\pi}{4} \)

The value of the integral in Eqn. (17) is computed with Levin's Transform, Gaussian Rule and the continued fraction Technique in [5] in Fig. (4). The file we called for C.F. is forward recursion algorithm is forwrec3.m in [5] which uses 1000 terms and having very low accuracy compared to the file gaussianTable.m for Gaussian Rule or LevinTransSum.m for Levin's Transform accelerating the series. All these Commands Window are in one figure Fig. (4) in appendix.

The reader should observed compute the series in Eqn. (20) for \( I(2,1) \) is given by
\[ I(2,1) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n-1} \quad \text{and} \]
\[ I(2,1) = \frac{\pi}{4}, \text{for } x = 1; \quad (23) \]

For such a series with terms having a variable \( x \), we apply LevinTranSumx.m as that require to submit an inline function \( f \)

having the ratio of terms and the variable \( x \) alone or submit a function handle with two variables the running indexing \( j \) for the terms and \( x \). Here we prefer the earlier choose. See Fig. (3) for its Matlab program.

**Table VII** Shows values for the three methods considered to compute \( I(1,1) \) in Eqn. (19) and Eqn. (21)

<table>
<thead>
<tr>
<th>Methods</th>
<th>C.F.</th>
<th>Gaussian</th>
<th>Levin</th>
</tr>
</thead>
<tbody>
<tr>
<td>values</td>
<td>1.441656663255249</td>
<td>0.693147157814239</td>
<td>0.693147180559931</td>
</tr>
<tr>
<td>Error ( \log(2) ) value</td>
<td>4.992509988754e-04</td>
<td>2.274570631843886e-08</td>
<td>1.432187701766452e-1</td>
</tr>
<tr>
<td>values</td>
<td>0.782898225889638</td>
<td>0.785398159934993</td>
<td>0.785398163397433</td>
</tr>
<tr>
<td>Error ( \frac{\pi}{4} ) value</td>
<td>0.00249993750781</td>
<td>3.46295525584026e-09</td>
<td>1.53210773982716e-14</td>
</tr>
</tbody>
</table>

**VII. COMPUTATIONAL REMARK**

The Error given in Table VII shows clearly that the continued fraction C.F. technique having the worst value for the approximation, even we have used about 1000 terms, due to growth of the nominator largely specially when we compute \( \frac{\pi}{4} \).

This one of the disadvantage of C.F. whenever the nominator accumulates to large number or the denominator approaches a very small number the precision will be lost and accumulated. So if one uses C.F. he must see the growth for both the nominator and the denominator first. However, it is known it is useful for the evaluations of Bessel's functions of first kind and the second kind. Gaussian rule, it is well known of its great advantages and easy to apply and whenever the number of points increased the number of decimals in accuracy increased. The algorithm Levin's Transform and other accelerating series like Euler's Transform are efficient and can compute to a very high accuracy, e.g. see the Error \( \frac{\pi}{4} \) value computed by Levin's in Table VII.

However, despite that it sometimes fails, I do astonish for this technique to compute
On Evaluations Some Sepcial Functions with Pade Approximant in Mupad Interface and Mathematica

\[ \eta(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - 2 + 3 - 4 + 5 \ldots (24). \]

The partial sum of this series is \( \{ S_n \} \) is \( S_0 = 0 \) and \( S_n = (-1)^n n \),
\[ n = 1, 2, 3, \ldots \]
Eqn.(24) shows it can't be summed just as \( \eta(0) \).

Levin's gives \( \eta(-1) = \frac{5}{4} = \zeta(-1) = \frac{1}{4} \)
but Levin's does it for \( \eta(-1) \) which gives an influence for suggestion to be more investigated and analyzed instantly within the commutation from the beginning \((\text{for from the first run it give the result } \frac{1}{4})\).

**APPENDIX**

\begin{verbatim}
a0=1;
f=inline('(-1)*1/(1+x)');
n=5;
LevinTranSum(f,n,a0)
format long
LevinTranSum(f,n,a0)
The Sum of the Series having 2*n+2 terms(1,1),(998,998)
1.000000000000000
0.500000000000000
0.333333333333333
0.583333333333333
0.783333333333333
0.616666666666666
0.759523809523809
0.634523809523809
0.745634920634921
0.645634920634921
0.736544011540112
0.653210678210678
The Sum of the Series using LEVIN TRANSFORM
using 2^n+2 terms
0.693452380952381
0.6933146935528455
0.693147179505577
0.693147180568758
0.693147180559931
log(2)-0.693147180559931
ans = 1.432187701766452e-14
\end{verbatim}

Fig.(4) Command Windows files LevinTranSum.m, forwrec3.m and quassianTable.m to evaluate the integral in Eqn.(15)

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**REFERENCES**

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