International Journal of Engineering Research And Management (IJERM)
ISSN : 2349- 2058, Volume-03, Issue-05, May 2016

On Evaluations Some Sepcial Functions with Pade Approximant in Mupad Interface and Mathematica

D.A.Gismalla

***Abstract*— The stability convergence for each method depends on the particular method chosen for that particular function to be approximated. However , it is well known that methods for approximations are classified in ascending order according to their efficiency and stability to do more better in approximation. These techniques in that order are PADE , CONTINUED FRACTION , CHEYBESHV and POWER SERIES . Despite the fact that Taylor's series are not so worst in general , many methods are developed for series acceleration as Euler's or Levin's Transformations. This means that an experienced numerical analysts can easily select which method is going to choose directly without runs into many complicated difficulty . So this the idea behind this paper is to compared some methods against the other in such away that the reader can be acquainted to know which method is better than other.**

Hence , this paper first give few commands in MUPAD INTERFACE and MATHEMATICA to get directly to approximate functions in a continued fraction or PADE form . second , a brief description for PADE is given with one of its important application in finding the root of equations. The Method is called Halley's algorithm that compared against Newton's method for finding the roots. Third , methods that we will consider are Gaussian quadrure , Pade Approximant , Continued Fraction and Levin's Transform to be compared and tested with examples showing which method is applicable and preferable than the other when they applied to a particular problem

I. COMMANDS IN MUPAD INTERFACE

In the Matlab Prompt command window the MUPAD interface will be invoked by just typing ***mupad*** followed with enter as **>>*mupad*** to get the prompt mupad as **[** . Now , type in the mupad promp **[**? followed with the command ***contfrac*** for seeking the syntax and the description command as a help information for Continued Fraction and then ***press enter*** as

[?contfrac

We will get all the information about the command ***contfrac*** for continued fractions.

Example 1 Continued Fraction Approximation

In the command command to express the Taylor's expansion for exp(x) around x=0 in a continued fraction form

>> contfrac(exp(x), x = 0)

Manuscript received March 09, 2016

D.A.Gismalla , Faculty of Mathematics and Computing ,Gezira University , P.O. Box 20 , Wad Medani , SUDAN

$$1+\cfrac{x}{1+\cfrac{x}{-2+\cfrac{x}{-3+\cfrac{x}{2+\cfrac{x}{5+O(x)}}}}}$$

>> contfrac(exp(-3*x^2), x = 0)

$$1+\cfrac{x^2}{-\frac{1}{3}+\cfrac{x^2}{-2+\cfrac{x^2}{1+O(x^2)}}}$$

Example 2 Series Approximation

>> ?series

We compute a Laurent expansion around the point x=1

>> s := series(1/(x^2 - 1), x = 1)

$$\frac{1}{2(x-1)}-\frac{1}{4}+\frac{x-1}{8}-\frac{(x-1)^2}{16}+\frac{(x-1)^3}{32}-\frac{(x-1)^4}{64}+O\left((x-1)^5\right)$$

>> series(x^(1/3)/(1-x),x)

$$x^{1/3}+x^{4/3}+x^{7/3}+x^{10/3}+x^{13/3}+x^{16/3}+O\left(x^{19/3}\right)$$

Example 3 Pade approximation

>> ?pade

Syntax

pade(f, x, <[m, n]>)

pade(f, x = x0, <[m, n]>)

Description

pade(f, ...) computes a Pade approximant of the expression f.

The Pade approximant of order [*m*, *n*] around $x = x_0$ is a rational expression

$$\frac{(x-x_0)^p\left(a_0+a_1(x-x_0)+\ldots+a_m(x-x_0)^m\right)}{1+b_1(x-x_0)+\ldots+b_n(x-x_0)^n}$$

approximating *f*. The parameters *p* and a_0 are given by the leading order term $f = a_0 (x - x_0)^p + O((x - x_0)^{p+1})$ of the series expansion of *f* around $x = x_0$. The parameters $a_1, \ldots, b_n$ are chosen such that the series expansion of the Pade approximant coincides with the series expansion of *f* to the maximal possible order.

The expansion points **infinity**, **-infinity**, and **complexInfinity** are not allowed.

If no series expansion of *f* can be computed, then **FAIL** is returned. Note that **series** must be able to produce a Taylor series or a Laurent series of *f*, i.e., an expansion in terms of integer powers of $x - x_0$ must exist.

The Pade approximant is a rational approximation of a series expansion:

>> f := cos(x)/(1 + x): P := pade(f, x, [2, 2])

$$\frac{-7x^2+2x+12}{x^2+14x+12}$$

For most expressions of leading order 0, the series expansion of the Pade approximant coincides with the series expansion of the expression through order $m + n$:

>> S := series(f, x, 6)

$$1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{13\,x^4}{24} - \frac{13\,x^5}{24} + O(x^6)$$

II. COMMANDS IN MATHEMATICA

Example 1 (power series approximation in Mathematica)

This gives a power series approximation to $(1+x)^n$ for x close to 0, up to terms of order x^3.

In[1]:=Series[(1+x)^n,{x,0,3}]

Out[1]=$1+n\,x+1/2(-1+n)nx^2+1/6(-2+n(-1+n)\,n\,x^3+O[x]^4$

Mathematica knows the power series expansions for many mathematical functions.

In[2]:=Series[Exp[-a t] (1+Sin[2 t]),{t,0,4}]

Out[2]=$1+(2-a)t+(-2\,a+a^2/2)t^2+(-4/3+a^2-a^3/6)t^3+1/2(32\,a-8a^3+a^4)\,t^4+O[t]^5$

Example 3 (Pade Approximant in Mathematica)

PadeApproximant[*expr*,{*x*,x_0,{*m*,*n*}}]

gives the Padé approximant to *expr* about the point $x=x_0$, with numerator order *m* and denominator order *n*.

PadeApproximant[*expr*,{*x*,x_0,*n*}]

gives the Padé approximant to *expr* about the point $x=x_0$, of order *n*.

Order [2/3] Padé approximant for Exp[*x*]:

In[1]:= PadeApproximant[Exp[x],{x,0,{2,3}}]

Out[1]= $$\frac{1+\frac{2x}{5}+\frac{x^2}{20}}{1-\frac{3x}{5}+\frac{3x^2}{20}-\frac{x^3}{60}}$$

PadeApproximant can handle functions with poles:

In[2] :=PadeApproximant[Exp[x]/x,{x,0,{2,3}}]

Out[2]= $$\frac{1+\frac{2x}{5}+\frac{x^2}{20}}{1-\frac{3x}{5}+\frac{3x^2}{20}-\frac{x^3}{60}}$$

III PADE TABLE FOR PADE APPROXIMANT

A function $f(z)$ is represented by a formal power series:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{l=0}^{\infty} c_l z^l,$$

where $c_0 \neq 0$, by convention. The (m, n)th entry $R_{m,n}$ in the Padé table for $f(z)$ is then given by

$$R_{m,n}(z) = \frac{P_m(z)}{Q_n(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}$$

where $P_m(z)$ and $Q_n(z)$ are polynomials of degrees not more than m and n, respectively. The coefficients $\{a_i\}$ and $\{b_i\}$ can always be found by considering the expression

$$f(z) \approx \sum_{l=0}^{m+n} c_l z^l =: f_{apx}(z)$$

$$Q_n(z) f_{apx}(z) = P_m(z)$$

$$Q_n(z)\left(c_0 + c_1 z + c_2 z^2 + \cdots + c_{m+n} z^{m+n}\right) = P_m(z)$$

and equating coefficients of like powers of z up through $m + n$. For the coefficients of powers $m + 1$ to $m + n$, the right hand side is 0 and the resulting **system of linear equations** contains a homogeneous system of n equations in the $n + 1$ unknowns b_i, and so admits of infinitely many solutions each of which determines a possible Q_n. P_m is then easily found by equating the first m coefficients of the equation above. However, it can be shown that, due to cancellation, the generated rational functions $R_{m,n}$ are all the same, so that the (m, n)th entry in the Padé table is unique. Alternatively, we may require that $b_0 = 1$, thus putting the table in a standard form.

Although the entries in the Padé table can always be generated by solving this system of equations, that approach is computationally expensive. More efficient methods have been devised, including the **epsilon algorithm**

If the difference of $Q_n(z)f(z) - P_m(z)$ having the first term with degree n+m+r+1 , for r>0 , then the rational function $R_{m,n}$ occupies $(r + 1)^2$ cells in the Padé table, from position (m, n) through position $(m+r, n+r)$, inclusive. The Pade Table is called normal for the function exp(x) that can be constructed using the MUPAD COMMAND as in the following Table I and it is not

normal for sin(x)-1 as in Table II The Pade approximant has many application in Physics and Mathematics for there much connection between it and the Continued Fraction Technique . It has a wide application specially in approximations and solving a system of non-linear equation as
Halley's algorithm emerged from it to find the roots of a polynomial.

Table I pade(exp(x) ,x,[n,m]) n,m=0(1)3

1	$-\frac{1}{x-1}$	$\frac{2}{x^2-2x+2}$	$-\frac{6}{x^3-3x^2+6x-6}$
$x+1$	$-\frac{x+2}{x-2}$	$\frac{2(x+3)}{x^2-4x+6}$	$-\frac{6(x+4)}{x^3-6x^2+18x-24}$
$\frac{x^2}{2}+x+1$	$-\frac{x^2+4x+6}{2(x-3)}$	$\frac{x^2+6x+12}{x^2-6x+12}$	$-\frac{3(x^2+8x+20)}{x^3-9x^2+36x-60}$
$\frac{x^3}{6}+\frac{x^2}{2}+x+1$	$-\frac{x^3+6x^2+18x+24}{6(x-4)}$	$\frac{x^3+9x^2+36x+60}{3(x^2-8x+20)}$	$-\frac{x^3+12x^2+60x+120}{x^3-12x^2+60x-120}$

Table II pade(sin(x)-1 ,x,[n,m]) n,m=0(1)4

-1	$-\frac{1}{x+1}$	$-\frac{1}{x^2+x+1}$	$-\frac{6}{5x^3+6x^2+6x+6}$
$x-1$	$x-1$	$\frac{5x-6}{x^2+x+6}$	$\frac{6(4x-5)}{x^3+6x^2+6x+30}$
$x-1$	$x-1$	$-\frac{x^2-6x+6}{x^2+6}$	$\frac{3(15x^2-34x+20)}{7x^3-3x^2+42x-60}$
$-\frac{x^3}{6}+x-1$	$-\frac{x^3}{6}+x-1$	$-\frac{7x^3+3x^2-60x+60}{3(x^2+20)}$	$-\frac{7x^3+3x^2-60x+60}{3(x^2+20)}$
$-\frac{x^3}{6}+x-1$	$-\frac{x^3}{6}+x-1$	$-\frac{7x^3+3x^2-60x+60}{3(x^2+20)}$	$-\frac{7x^3+3x^2-60x+60}{3(x^2+20)}$

IV. SOLVING KEPLER'S EQUATION USING PADE'S APPROGOXIMANTS

In order to demonstrate the application of the Pade' approximants to a problem revelant to Astronomy , consider Kepler's equations

E-esin(E)=M , M$\in (0, 2\pi)$, e$\in [0 , 1]$

which yields $f(x) = x - e\,sin(x)\_M$ and

$f'(x) = 1 - e\cos(x),\ \ f''(x) = e\sin(x)$

It has been shown in [1], pp. 24 , the correction Δx_i derived from the pade approximant of order (1 ,1) is

$$\Delta x_i = -\frac{ff'}{(f')^2 - \frac{1}{2}f f''} \quad ,i = 0,1,2,.. \qquad (1)$$

Eqn.(1) is called Halley's algorithm for finding a root from a non-linear equation.
Now , Halley's algorithm is similar to Newton's iteration technique given by

$$\Delta x_i = -\frac{f}{f'} \quad .i = 0 .1 .2. .... \qquad (2)$$

where Δ *is forward difference opearater an*

$\Delta x_i = x_i - x_{i-1}$

Both the techniques can be used to solve a system of non-linear equations provided that the initial starting guess solution x_0 is given . Now to find the root for Kepler's Eqn.(3) ,We write for Halley's algorithm

$$\Delta x_i - \frac{(x - e\sin(x) - M).(1 - e\cos(x))}{(1 - e\cos(x))^2 - \frac{1}{2}(x - e\sin(x) - M).e\sin(x)} \quad ,i = 0,1,2,3 \qquad (3)$$

while we write for Newton's Raphson Method Eqn.(4)

$$\Delta x_i = \frac{(x - e\sin(x) - M)}{.(1 - e\cos(x))} \quad ,i = 0,1,2,3 \qquad (4)$$

It is well known that Newton's Method fails whenever the initial starting solution is far from the real exact solution and since Halley's algorithm has the order of convergence as Newton it may diverge similarly. Now with M=0.6 , e=0.9 and x_0 =0.08 , the result solution for both method is x=1.497589413390409

V. EVALUATION ON SOME SPECIAL FUNCTIONS

Debye function A.

The idea for stability and efficient approximating result is achieved when a suitable method is applied to a particular function . Here , for example , We apply Gaussian rule , Pade approximant and then integrate with Gaussian rule , on Debye function given in [2] ,page (998) which is defined as

$$\int_0^\infty \frac{t^n}{e^t - 1} dt = n!\,\zeta(n+1) \qquad (5)$$

where n! means factorial n while $\zeta(n+1)$is the Zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^{-s}} = \prod_p (1 - p^{-s})^{-1} \qquad (6)$$

where $\prod_p (product\ over\ all\ prime\ numbers\ )$

In [2] , the hand book of mathematical functions edited by Milton and Stegun a table evaluating the Debye function from x=0 to 10 and n=1 to 4 to evaluate the integral

$$\int_0^x \frac{t^n}{e^t - 1} dt \qquad (7)$$

with no hint describing the methods they used but certainly with a computer of high accuracy for decimal places. We ,evaluate it for x=0.1 and x=10 while n=1 first and second n=4. The procedure that We adopted is Gaussian Quadratue with five nodes as in Fig.(1) for its Matlab program. The next approach that We applied when We get advantage of the Pade command in MUPAD interface to approximate the integrand function in Eqn. (7) by a rational pade(n,m) and then Gaussian Rule again with five nodes is applied. The result are collected from command window figures as in Fiq.(2) but some values are obtained when apply the command Pade first and then We apply Gaussian Rule ((without given the command window here but cited the result only)) and then all are inserted in Debye function Table III and Table IV.

```
function [Ie]=quassianTable(f,a,b,n)
% Example  Integration using  Guassian Quatrature rules
  % In Matlab command window
  % syms x;
  % f=inline('x^4 /(exp(x)-1)');
  % a=0; b=0.5 ;
  % n=5 ; % Guassian-Table is given and n runs from 2 to 5
  % [Ie]=quassianTable(f,a,b,n)

   c=[1.0000000000  0.5555555556  0.3478548451   0.2369268850
      1.0000000000  0.8888888889  0.6521451549   0.4786286705
      0.0000000000  0.5555555556  0.6521451549   0.5688888889
      0.0000000000  0.0000000000  0.3478548451   0.4786286705
      0.0000000000  0.0000000000  0.0000000000   0.2369268850];
  x=[ 0.5773502692   0.7745966692  0.8611363116   0.9061798459
    -0.5773502692  0.0000000000  0.3399810436   0.5384693101
      0.0000000000 -0.7745966692 -0.3399810436   0.0000000000
      0.0000000000  0.0000000000 -0.8611363116  -0.5384693101
      0.0000000000  0.0000000000  0.0000000000  -0.9061798459];

  sum=0;
  for j=1:n
    t=((b-a)*x(j,n-1)+a+b)/2;
     sum=sum+c(j,n-1)*feval(f,t)*(b-a)/2;
  end
  Ie=4*sum /b^4 ;
```

Fig.(1) qaussianTable.m file for qadrature

sym x ;
a=0; b=10 ;
n=5 ;
f=inline('x/(exp(x)-1)');
[Ie]=quassianTable(f,a,b,n)
 	
Ie = 0.1644
b=0.5;
[Ie]=quassianTable(f,a,b,n)
 	
Ie = 0.8819
	syms x;
	f=inline('x^4 /(exp(x)-1)');
	a=0; b=0.5 ;
	n=5 ;
	[Ie]=quassianTable(f,a,b,n)
	
	Ie = 0.8138
	b=10;
	[Ie]=quassianTable(f,a,b,n)
	
	Ie = 0.0097

Fig.(2) Gaussian for Debye function when n=1&4 , x=0.1 and x=10

Table III Evaluation of Debye function when n=1 , x=10 and x=1

x	Apply Gaussian only $\int_0^x \frac{t}{e^t - 1} dt$	Apply Pade and Gaussian $\int_0^x \frac{t}{e^t - 1} dt$
0.1	0.8819	0.8819
10.0	0.1644	0.1644

Table IV Evaluation of Debye function when n=4 , x=0.1 and x=10

x	Apply Gaussian only $\int_0^x \frac{t^4}{e^t-1}dt$	Apply Pade & Gaussian $\int_0^x \frac{t^4}{e^t-1}dt$
0.1	0.8138	0.8138
10.0	0.0097	0.0134

The reader should observed that the values for integral in the second column in Table III and Table IV are obtained when n=1 with command Pade approximant as
pade(x/(exp(x)-1), x,[4,8])

$$\frac{15120\,(x^4-40\,x^3+660\,x^2-5280\,x+17160)}{x^8+45\,x^7+1080\,x^6+17640\,x^5+211680\,x^4+1890000\,x^3+13305600\,x^2+49896000\,x+259459200}$$

while for n=4 with command Pade approximant as
pade(x^4/(exp(x)-1) , x ,[4,5]) (8)

$$\frac{30\,x^3\,(x^4-28\,x^3+336\,x^2-2016\,x+5040)}{x^5+30\,x^4+420\,x^3+5040\,x^2+15120\,x+151200}$$

Now ,We apply Gauusian Rule with 5 points to get Ie=0.013 which is the worst result of the four results given in Table III and Table IV. This certainly due to the accumulation of rounding errors and the precision of our evaluation is not high and Gaussian Rule can't attained higher accuracy more than 9 digits of places. Also , We observe the order in the command in Eqn.(8) is Pade[4,5] but the Pade Approximant is of order 8 only. All these will effect the approximation results. Even , here there is another plenty one must adjust the file quassianTable.m given in Fig.(1) for the denominator and numerator are two long to be submitted as parameters easily. Lastly, both these techniques are very important and each one is suitable for certain problems, e,g. Pade can be designed for problems having poles or singularity inside the domain or at its end points.

B. The Zeta and Eta Functions

Next , as a typical example of conditional convergent series , the eta function $\eta(z)$

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \quad , \quad \mathcal{R}e(z) > 0 \quad (9)$$

Which is connected to Riemann's zeta function given by

$$\eta(z) = \left(1-2^{(1-z)}\right)\zeta(z) \quad , \quad z \neq -1 \quad (10)$$

where Riemann's zeta

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad , \quad \mathcal{R}e(z) > 1 \quad (11)$$

or , alternatively by

$$\left(1-2^{(1-z)}\right)\zeta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \quad , \quad \mathcal{R}e(z) > 0 , z \neq -1 \quad (10)$$

A series which converge for $\sigma > 0$, i.e. also in the strip $0 < \sigma \leq 1$

By use of more advanced methods the following relation can be hold

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right)\Gamma(1-s)\zeta(1-s) \quad (11)$$

through this relation Eqn.(11) , the function can be computed when $\sigma > 0$. Alternatively useful representation is given by

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{x^n}{e^x-1}dx \quad , \sigma > 1 \quad (12)$$

The formula in Eqn.(12) can be transformed to

$$\zeta(s) = \frac{e^{-i\pi s}\,\Gamma(1-s)}{2\pi i}\int_C \frac{z^{s-1}}{e^z-1}dz \quad (13)$$

where C is the real axis from ∞ to ε , the circle $|z| = \varepsilon$, and again the real axis from ε to ∞. Eqn.(13) implies that when s=0 can be written as

$$\zeta(0) = \frac{1}{2\pi i}\int_C \frac{1}{z^2}\left[1-\frac{z}{2}+\cdots\right]dz = -\frac{1}{2} \quad (14)$$

Alternatively , instead of using contour integration ,the divergent series for the Eta function

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \Rightarrow \eta(0) = 1 - 1 + 1 - \cdots (15)$$

Hence , using suitable manipulating We find that $\eta(0) - \frac{1}{2} \to \zeta(0) - -\frac{1}{2}$

A very nice discussion can be found in [3] from which the zeros and the calculated values for $\zeta(s)$ as , We cited some from page 113 to be

$\zeta(-2m) = 0$,

$$\zeta(1 - 2m) - \frac{(-1)^m B_m}{2m}$$

$$and \zeta(-2m) = \frac{1}{2} \frac{B_m (2\pi) 2^m}{(2m)!} \quad (16)$$

where B_m , m=1,2,3, and -2 ,-4 ,-6… are trivial zeros of zeta function .

From [3] , We cited some numerical values for the zeta functions as in Table V exactly as they are and then We evaluate these values again to be compared with our results. We apply Levin's Transform in [6] with its program MATLAB as in Fiq.(4.7) ,pp27 . In fact We evaluate the Eta function given by Eqn.(9) and simultaneously from the result of the program the Zeta function is evaluated by Eqn.(10) . Table VI shows the values of Eta and Zeta where the number of terms is 12 and the number of decimal places for accuracy is at least 10. Observes that when s=-1
Lenin's Transform converge 0.25 which is the value for Eta function when s=-1 which gives the value for Zeta function $\zeta(s)$ =- 1/12 and . $\zeta(0)$ =- 1/ 2

Table V Zeta values We cited from [3]

S	$\zeta(s)$
5/2	1.3414872573
3/2	2.6123753487
4/3	3.6009377506
1/2	-1.4603545088
1/3	-0.9733602458
0	-0.5
-1/2	-0.2078862250
-1	-0.083333333

Table VI Values We Computed using Levin for Eta &Zeta

n	s	$\zeta(s)$	$\eta(z)$
12	5/2	1.341487257250917	0.867199889012184
12	3/2	2.612375348685454	0.765147024625398
12	4/3	3.600937750458803	0.742871563883650
12	1/2	-1.460354508809559	0.604898643421619
12	1/3	-0.973360248350768	0.571752833825269
	0	-0.5	0.5
12	-1/2	-0.207886224977360	0.380104812609694
	-1	-1/12	0.25

```
function   LevinTranfSumx(f,n ,a0 ,x)

% Levin Transform for Series Summation with each term having variable x
% This Technique is a series Technique for which the integral is expressed
% as a series FIRST and then LEVIN is applied. The general term for the
% series can be expressed as an inline function or a handle object with
% the initial term submitted to the program in advanced to generate the
% other terms with the number of terms n to be taken for the sum.
% The function f in LevinTranfSum(f,n ,a0 ,x) is a ratio  to generate other terms .
% In Matlab command window
% syms s;
% f=inline(' (-1)^s*(2*s-1)/(2*s*(2*s+1))');
% a0=x ;
% n=5;
 % LevinTranfSumx(f,n ,a0 ,x)
 global UT ;
   for k=1:n
      [S, UT]=LevinTransformx(f,k ,a0 ,x);
       F(k)=UT;
   end
     disp(' The Sum of the Series having  2*n+2 terms')
     disp([ S']);
     disp(' The Sum of the Series using LEVIN TRANSFORM using 2* n+2  terms')
     disp([ F']);
function  [ S , UT]=LevinTransformx(f, k ,a0 ,x)
   a(1)=a0 ;
   S(1)= a(1) ;
   C(1)= 1;
 TotalSumDen(1)=1;
 TotalSumNum(1)=1;
for j=1:2*k+1
     a(j+1)=feval(f,j)*x^2*a(j);
     S(j+1)=S(j)+a(j+1);
     C(j+1)=(2*k+2-j)*C(j)/(j);
     TotalSumDen(j+1)=TotalSumDen(j) +
(-1)^j*C(j+1)*(j+1)^(2*k-1)*S(j+1)/a(j+1);
     TotalSumNum(j+1)=TotalSumNum(j) + (-1)^j*C(j+1)*(j+1)^(2*k-1)/a(j+1);
end
     UT= TotalSumDen(2*k+2)/TotalSumNum(2*k+2);
```

Fig.(3) File LevinTranfSum.m for series with terms having variable x

VI. TRANSFORM INTEGRALS TO CONTIUED FRACTION

If , we consider the integral

$$\mathrm{I(n,m)} = s = \int_0^1 \frac{x^{m-1}}{1+x^n}\,dx \qquad (17)$$

and after integrating with x=1 , the value s will be

$$s = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3m} + \cdots \qquad (18)$$

It can be shown from theorem(2.1) in [5] the Continued Fraction is

$$\frac{1}{s} = m + \cfrac{m^2}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{\ddots + \cfrac{}{n + \text{etc}}}}} \qquad (19)$$

Further , if We again consider the integral in Eqn.(17) when m=1 and n=2 ,its value actually $\frac{\pi}{4}$ when x=1 and the integral for general x is of the form in Eqn.(20)

$I(2,1) = s = \int_0^x \frac{x^{m-1}}{1+x^n} dx \qquad (20)$

and it can be represented as C.F in Eqn.(21) such as

$$I(2,1) = \frac{1}{x+}\frac{1^2x^2}{3-x^2+}\frac{3^2x^2}{5-3x^2+}\cdots\cdots \frac{(2n-3)^2x^2}{(2n-1)-(2n-3)x^2}\ldots.(21)$$

Now if We put x=1 ,We get as in [5] , the C.F. for $\frac{\pi}{4}$ to be

$$\frac{\pi}{4} = \frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\cdots\frac{(2n-3)^2}{2}\ldots.(22)$$

Hence , the computed value of the integral in Eqn.(17) for m=n=1 is in Table VII having the exact value I(1,1)=log(2) while the computed value of the integral in Eqn.(17) for m= 1 and n=2 is in Table VII having the exact value I(2,1)= $\frac{\pi}{4}$

The value of the integral in Eqn.(17) is computed with Levin's Transform , Gaussian Rule and the continued fraction Technique in

[5] in Fig.(4) .The file We called for C.F. is forward recursion algorithm is forwrec3.m in [5] which uses 1000 terms and having very low accuracy compared to the file quassianTable.m for Gaussian Rule or LevinTranfSum.m for Levin's Transform accelerating the series . All these Commands Window are in one figure Fig.(4) in appendix.

The reader should observed compute the series in Eqn.(20) for I(2,1) is given by

$$I(2,1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n-1}}{2n-1} \quad and$$

$$I(2,1) = \frac{\pi}{4} \quad , for\ x = 1; \quad (23)$$

For such a series with terms having a variable x , We apply LevinTranfSumx.m as that require to submit an inline function f having the ratio of terms and the variable x alone or submit a function handle with two variables the running indexing j for the terms and x. Here We prefer the earlier choose.

See Fig.(3) for its Matlab program.

Table VII Shows values for the three methods considered to compute I(1,1) in Eqn.(19) and Eqn.(21)

Methods	C.F.	Gaussian	Levin
values	1.441656663255249	0.693147157814239	0.693147180559931
Error $log(2)$ – value	**4.992509988754e-04**	**2.274570631843886e-08**	1.432187701766452e-14
values	0.782898225889638	0.785398159934493	0.785398163397433
Error $\frac{\pi}{4}$-value	**0.002499937507810**	3.462955255884026e-09	**1.532107773982716e-14**

VII. COMPUTIONAL REMARK

The Error given in Table VII shows clearly that the continued fraction C.F. technique having the worst value for the approximation ,even We have used about 1000 terms, due to growth of the nominator largely specially when We compute $\frac{\pi}{4}$. This one of the disadvantage of C.F. whenever the nominator accumulates to large number or the denominator approaches a very small number the precision will be lost and accumulated. So if one uses C.F. he must see the growth for both the nominator and the denominator first . However , it is known it is useful for the evaluations of Bessel's functions of first kind and the second kind. Gaussian rule ,it is well known of its great advantages and easy to apply and whenever the number of points increased the number of decimals in accuracy increased . The algorithm Lenin's Transform and other accelerating series like Euler's Transform are efficient and can compute to a very high accuracy , e.g .see the Error=$\frac{\pi}{4} - value\ computed\ by\ levin's$ in Table VII.

However ,despite that it sometimes fails , I do astonish for this technique to compute

$$\eta(-1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{-1}}=1-2+3-4+5\ldots(24).$$

The partial sum of this series is $\{S_n\}$ *is* $S_0=a_0=1$ and $S_n=(-1)^n\,n$,
$n=1,2,3,\ldots$

Eqn.(24)shows it can't be summed just as $\eta(0)$

Levin's gives $\eta(-1)=\frac{1}{4}\Rightarrow\zeta(-1)=-\frac{1}{12}$
but levin's does it for $\eta(-1)$ which gives an influence for suggestion to be more investigated and analyzed instantly within the commutation from the beginning ((for from the first run it give the result $=\frac{1}{4}$.))

APPENDIX

```
a0=1;
f=inline('(-1)*1/(1+1/s)');
n=5;
LevinTranfSum(f,n ,a0)
  format long
LevinTranfSum(f,n ,a0)
 The Sum of the Series having  2*n+2 terms
   1.000000000000000
   0.500000000000000
   0.833333333333333
   0.583333333333333
   0.783333333333333
   0.616666666666666
   0.759523809523809
   0.634523809523809
   0.745634920634921
   0.645634920634921
   0.736544011544012
   0.653210678210678
The Sum of the Series using LEVIN TRANSFORM
using 2* n+2  terms
   0.693452380952381
   0.693146595528455
   0.693147179505577
   0.693147180568758
   0.693147180559931
log(2)-0.693147180559931

ans = 1.432187701766452e-14
```

```
n=1000;
 for k=1:n
   g(k)=1;
   f(k)=(1+k-1)^2;
 end
 [y , k] = forwrec3(f , g  ,n ,  1e-10)

y = 0.441656663255249
k =  1000

log(2)-1/1.441656663255249

ans = -4.992509988754890e-04
```

```
syms x;
f=inline('1 /(1+x)');
n=5;
a=0; b=1;
[Ie]=quassianTable(f,a,b,n)

Ie =    0.693147157814239

log(2)- 0.693147157814239

ans =  2.274570631843886e-08
```

Fig.(4) . Command Windows files LevinTranfSum.m , forwrec3.m and quassianTable.m to evaluate the integral in Eqn.(15)

ACNOWLEDGEMENT

I would like to thank Taif University ,Saudi Arabia to be appointed Professor in the Department of Mathematics ,Riana College , during 2012-2016 for which some accomplished research papers are published

REFERNCES

[1] Josef Kallrath Basf OSEF KALLRATH On Rational Function Techniques and Pade Approximants , *An Overview ,AG ZX/ZC – C13, D-67056 Ludwigshafen, GERMANY e-mail: kallrath@zx.basf-ag.de* September 16, 2002

[2] Abramowitz,M ,and Stegun ,I.A. 1964 , Handbook of Mathematical Functions , Applied Mathematics Series , Volum 55 (Dover Publication New York)

[3] Clark-Erik Frberg , Numerical Mathematics ,Theory and Computer Application. The Benjamin /Cummings Publishing Company ,Inc , 1985.

[4] D.A.Gismalla , Lynne D. Jenkins and A.M.Cohen Acceleration of Convergence of Series for Certain Multiple Integrals , I.J.C.M,Vol. 24, pp 55-68, 1987.

[5] D.A.Gismalla, Survey on Transformations for Infinite Series to Continued Fractions with Matlab Program for Computing some. International Journal Innovations for Engineering Research and Management . vol.3, May 2016 , India.

[6] D.A.GISMALLA , Computer Oriented Programs on Multi-Dimensional Numerical Integration , International Journal of Engineering and Technical Research (IJETR) ISSN: 2321-0869, Volume-2, Issue-2, February 2014