A Noval Study of Domination Parameters in Graph Theory

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Abstract— **There is a rapid growth in study of domination in graphs in last four decades. Social network problems wireless communication networks, Cyber Security issues, biological model networks, service location problems make use of domination concept. This paper presents a detailed survey on fundamentals of dominations related parameters, which gives an idea about the rapid growth and development of the concept to the readers / students at a beginning level.**

Index Terms— **Edges, Vertices, induced subgraphs, graphs, connected graphs.**

I. INTRODUCTION

Graph theory has rapid growth in different sectors including Science and Technology. Communication networks can be investigated using techniques of graph theory, a graph is a mathematical representation of a given network. In any network a graph represents a relationship between the objects having features in common. Puzzle were a source of inspiration for origin as well as development of graph theory. ***"Leonhard Euler"*** is considered as father of graph theory. He solved the puzzle of ***"Konigsberg bridge problem"*** in 1736 and published a paper in graph theory. The results for the puzzle made way for a new class of graphs known as ***"Eulerian graphs".*** Kirchhoff, a known physicist developed the ***"trees"*** concept to solve equations of electric current in 1847. The electrical networks topology could be efficiently represented by ***"spanning trees"*** and hence can be used to solve simultaneous electrical current equations. ***"Arthur cayley",*** a renowned chemist, discovered "Trees" in 1857 one of his investigations. He was investigating the number of structural isomers present in saturated hydrocarbons **($C_K H_{2K+2}$).** This concept of tree finds application in real time like binary search, mails sorting techniques etc.

The popular puzzle game "Around the World" which was invented by ***"W.R Hamilton"*** in 1959 made is popular as "***Travelling salesmen problem (TSP)".*** However there is no efficient algorithm to find out whether a graph which is given is Hamiltonian or not.

The growth of graph theory is aided by operational researches. Many practical problems like maximal network flow present in a graph, minimum spanning tree present in a graph, shortest distance existing between two nodes etc., have been solved using graph theory concepts. ***"Ford & Fulkerson"*** proposed and solved the maximal flow problem in 1956. This plays an important role in solving any kind of transportation network.

The planarity concept was initiated by ***"Euler"*** and developed by ***"Kuratowski"*** in the year 1930. The electrical and road networks have adopted planarity notion. The life space of a human can be represented by a planar graph, as proposed by the known psychologist ***"Lewin"*** in 1936. A planar graph has its regions that resemble an atlas. According to the ***"Four Colour Conjecture"*** an atlas can be coloured using four colours. This was proposed by Mobius in the year 1840 when he attempted to colour the map of England's countries and still remained a conjecture. During the study of problems related to colouring, many research related areas like covering, chromatic polynomial, chromatic number, partitioning, matching found considerable contribution.

To solve certain problem like logic partition of digital computers , coding theory , reducing number of states of sequential machines , the partition of vertices or edges present in a graph is considered more details about above discussions can be found in Harary [1]. ***"Narsingh Deo"*** has treated graph theory concepts from computer science and engineering point of view.

II. ORIGIN AND DEVELOPMENT OF DOMINATION THEORY

There has been a rapid growth in the study of the concept of domination in graphs in past few years. This area of graph theory is found to be one of the prominent areas of research and as a proof hundreds of research articles have been published in the recent years covering various applications of dominations theory. Domination concept can be explained as follows: Most of the networks like traffic (road), communication and organizational networks make use of a set of nodes capable of controlling the rest identification of a smallest size of controlling nodes is necessary to reduce manpower and resources . In terms of graph theory, this problem is like finding a smallest subset ***'A'*** having vertex set ***'V'*** for a given graph ***'G'*** in such a way that every vertex ***'v € V'*** is either a member in ***'A'*** or has a neighbour present in ***'A'*** hence it can be stated that ***'A'*** dominator vertices belonging to graph ***'G'*** and is thus called ***"Dominating set"*** of graph ***'G'***. Dominating number of a graph ***'G'*** is actually the size of the smallest dominating set and is denoted by **'s (G)'.** Indians invented a board game called 'chess' in the medieval period of i millennium AD and this played a key role bind the origin of domination theory graphs. The chess fanciers from Europe pondered over determination of minimum number of queens need in attacking each and every square present in the chess board. This was popularly known as queen's problem. The solution for this was rightly to be five. Survey in literature shows that ***"Jaenisch"*** was the first person to have studied the queen's problem with mathematical approach and has published an article on it. Three type of queen's problem were described by ***"W.W Rouse Ball"*** in 1982 relating to domination independent domination and N-queen's problem (in which queen's attack all squares except themselves). The study further and put forth successful results for "Bishop king" knight and rook chess pieces in 1964. In 1952 cloude bengge [2] in his book has dealt the domination concept and has introduced the term "co-efficient of external stability" to refer the graph's domination number ***"oystein ore"*** [3] published a book in 1962 in which he has made a study on

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domination concept and introduced the terms ***" Domination number"*** and ***"Dominating Set"*** having notation "**A(G)".**
A survey paper on domination was published by ***"Cockayne and Hedetniemi"* in 1977.** They made use of the symbol s (G) to represent domination number for a given graph G, "**Haynes et al.,"** [4] has also written a book on growth and development of domination concept in graphs.

III. VARIETY OF PARAMETERS IN DOMINATION CONCEPT

There are various types in domination. The authors of [5] cite that more than 75 varieties of domination parameters exist. Real time applications make use of any one of the domination type. Different type of dominating sets can be formed by putting an extra condition either on domination method or on a dominating set.
To systemize different parameters of domination, Harary and Haynes [6] a definition for conditional domination number s (G: P). accordingly, it is defined as the minimum cardinality of a dominating set **'A □□V'** such that the induced subgraph (A) satisfies the property P. Few properties imposed on **'A'** are :

PROPERTY 1: THE INDUCED SUBGRAPH HAS NO ISOLATED VERTICES.

"Cockayne et al" introduced this parameter [7] and accordingly it is termed as total dominating set. Its number **'s_1 (G)' is** the minimum cardinality of such a set.

PROPERTY 2: THE INDUCED SUBGRAPH HAS NO EDGES.

Such a domination is termed as independent domination is termed as independent domination. The independent domination number ***'i (G)'*** is defined as the minimum cardinality of an independent dominating set. ***'A'*** is an independent this parameter has been studied in detail by ***"Allan and Laskar"*** [8] and others.

PROPERTY 3: THE INDUCED SUBGRAPH IS COMPLETE

Dominating clique is the term given to the dominating set that corresponds to this property. Its number is denoted as **'s_{cl} (G)'.** A study on a dominating clique and its properties was made by ***"Cozzens and Kelleher".***

PROPERTY 4: THE INDUCED SUBGRAPHS IS CONNECTED

Such a domination is termed as a connected domination in graph theory. This was introduced by ***"sampath kumar"*** and **"walikar"** [9] its number is denoted **as 's_c (G)'.**

PROPERTY 5: THE INDUCED SUBGRAPH IS A SET OF INDEPENDENT EDGES

A dominating set having this property is termed as induced-paired dominating set and its number is termed **'s_{ip} (G)'.** This concept has been introduced by ***"Studer et al."*** [10].It can be observed that ***'s (G)' ≤ 's(G:P)'*** for any ***'P'***. It should be noted that all conditional domination parameters do not exist for any given graph ***'G'***. For instance, if ***'G'*** has isolated vertices, then property 1 does not hold good. If ***'G'*** is a disconnected graph, property 4 does not hold good.

The complementary dominating set ***'V-A'*** can be treated according to its inherited properties. Keeping this perspective in view, following are the few dominating sets '***A'*:**

STRONG SPLIT DOMINATING SET:

The induced subgraph **'< V- A>'** is actually a null graph. The symbol used to denote its number is **'s_{rr} (G)'.**

CO-TOTAL DOMINATING SET:

The induced subgraph **'<V-A>'** does not possess isolated vertices. Its number is denoted by the symbol **'s_{cot} (G)'**

SPLIT DOMINATING SET:

The induced subgraph **'<V-A>'** is a disconnected one and its number has a symbol **'s_r (G)'.**

NON-SPLIT DOMINATING SET:

The induced subgraph **'<V-A>'** is a connected one and its number is symbolized by **'s_{nr} (G)'.**
The split, non-spilt and strong split dominating sets are introduced ***"Janakiram and Kulli"*** [11][12][13] and co-total dominating set was introduced by ***"kulli et al"*** [14]. ***"Domke et. al"*** [15], has independently introduced the co-total domination and termed as restrained domination in graph. Kulli and his researcher's teams have combined the above listed properties and imposed the conditions of 'A ***to V-A'*** and formed more domination parameters. Some more domination n parameters available in literature are listed below:
For any given graph ***'G'***, the vertices and edges are treated as its elements. A set ***'X'*** containing elements of ***'G'*** is termed as ***"Entire Dominating Set"*** provided each element, not present in ***'X'*** is either adjacent to any element in ***'X'*** or incident to atleast one element in ***'X'***. The above parameter of domination was introduced by "***Kulli et al"*** [16]**.**

a. For a graph ***'G'*** and its complement graph ***'G'***, a dominating set ***'A'*** will be a global dominating set if ***'A'*** is dominating set to both ***'G'*** and ***'G' .***
b. ***"Fink and Jacobson"*** [17] introduced the concept of multiple domination in graphs. Given a graph ***'G',*** a dominating set ***'A'*** is said to be k-dominating set, if every vertex in ***'V-A'*** is dominated by at-least k vertices of **'A',** i.e **|N (v) | □□D | ≥ k.** The number has a symbol ***'s_k (G)'.***
c. ***"Sampath kumar and pushpalatha"*** have developed another domination parameter termed 'point set domination'. If a subgraph **S= {v},** then a set-dominating set becomes point set domination.
d. ***"Sampath kumar and pushpalatha"*** introduced and studied a set-dominating set of a graph [18]. A set domination in graph is one in which domination between sets replaces domination between vertices.

"Haynes et al" [19] has authored a book which gives details about graphs and its domination parameters. Kulli, the Indian graph theorist has entered the details of various domination parameters and its results in the book he has authored [20]. Domination concept can be applied to edges of the graph in addition to its application to vertices. Most of the domination parameters related to vertices can also be equivalently applied to edge related domination parameters.
Edge Dominating Set: A set containing edges ***'X □□E',*** is termed edge dominating set if the edges that are not present in 'X' are adjacent to some edge in **'X'.** The cardinality of a smallest edge dominating set of **'G'** is called edge. Domination number for the given graph it is given the symbol **'s^1 (G)'.**

"Gupta" studied the line/edge domination [21] in 1969 as line-line covering having symbol ***'α_{11} (G)'*** for the line-line covering number. The tress parameter [22] was studied an edge domination by ***"Mitchell and Hedetniemi"***. In 1998, various parameters of notion like total, connected, independent edge dominating sets were studied by "***Armugam and Velammal"*** [23].

"Gayathri and Mohanaselvi" have rendered their contributions to domination related edge analogues to the field of study since 2008. The focus of their conditional edge domination parameters is on the properties of co-edge dominating set ***'E\X'***. Some of the contributions include co-edge maximal domination [24], co-edge tree/spanning tree domination [25], co-edge split/ Non-split domination, co-edge complete/ independent / total domination. An exclusive treatment of these parameters is done [26].

Domination parameter are discussed and characterized for properties and bounds for various graphs. A study is made for NP- completeness nature. Study is also carried on their applicative nature in various fields, ***"Cockayne and Hedetniemi"*** have conducted a study on domatic partition of graph [27]. The D-partition of a graph is defined as a partion of vertices to dominating sets. Domatic number of the graph is defined as maximum order of such a partition. Extension of this notion is made to other domination parameter using suitable tags like total, connected, independent etc.

IV. Application of Domination Parameters

There are a number of real life application for domination in graphs. All parameters have one or more real life applications. Some of the applications of domination parameters from literature are listed below.

Telecommunication area is one of the key application area of domination parameter. ***"Berge"*** has given information about application of domination parameter for ***"Survellance"*** purpose to locate radar stations [28] Domination parameters can be use in commination networks for mounting the transmission stations as discussed by ***"Liu"*** [29]. Domination parameters find extensive use in police stations, post offices, schools, fore stations etc.

"Kalbeisch et al" [30] ***"Cockayne"*** and ***"Hedetniemi"*** [31] have discussed the domination application in coding theory. If a n-dimensional vector codes is modeled as graphs of hypercubes (of n-cubes) or analogous graphs, then (n,p) covering sets, single error correcting codes , perfect covering sets have all graphs with dominating sets with extra properties. Domination parameter was used by Haynes et al, to distinguish secondary Ribonucleic Acid motifs from other contents in biological models [32]. A set of protein cells is topologically central in human PPI network from a minimum dominating set of PPI network. This was explored by ***"Milenkovic et al"*** [33]. The authors found that drug targets, signalling pathways, biologically central genes were all conquered by dominating sets. An ad-hoc network is a wireless communication system having no fixed infrastructures. The idea that a connected dominating set resembles a virtual backbone of ad-hoc networks (finds applications in routing of messages) was put forth by ***"Jie Wu and Hailan Li"*** [34], ***"Das and Bhargavan"*** [35]. Domination as stated finds application in so many life instances. Domination concept finds application in land surveying where the number of location get minimized for a surveyor to carry out surveying measurements. A set of representatives can be selected from employees list in any organisation using domination concept. A minimum number of processors can be selected to receive information and data from other processors in computer communication networks using domination concept. A public transport (ex: bus) can use domination to find optimal route.

Conclusion

Domination concept has huge number of applications is real life. Hence this field of research has attracted many students and researchers. This paper is an attempt to present some basic knowledge of graph theory and domination in graph. The presentation covers the origin, growth, types and applications of domination.

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