An Adjustable State Estimator for Discrete-Time Nonlinear Stochastic Systems

Sie Long Kek, Kim Gaik Tay, Sy Yi Sim

***Abstract*—In this paper, an adjustable state estimator is introduced for estimating the dynamics of a discrete-time nonlinear system that is disturbed by a sequence of random noises. In our approach, the Kalman filtering theory, which is employed to estimate the dynamics of the linear system, is completed with adding the adjusted parameters into the model used. In this way, the differences between the real plant and the model used can be measured iteratively. On the other hand, the output, which is measured from the real plant, is fed back into the model used in order to update the sequence of the optimal state estimate in each iteration step. When the convergence is achieved, it is found that the iterative solution approaches to the true optimal solution of the original estimation problem in spite of model-reality differences. In additional, the convergent property of the adjustable state estimator is also given. For illustration, three examples are studied and the results obtained show the efficiency of the approach proposed.**

***Index Terms*—Adjustable state estimator, iterative solution, Kalman filtering theory, model-reality differences.**

I. Introduction

Estimating the state dynamics accurately from a nonlinear dynamical system that is disturbed by Gaussian white noise sequences is a challenging task. This estimation is subject to the fluctuation behavior appeared in the dynamical system that gives an unpredictable response, and makes the dynamical system even more complex. In this point of view, the Kalman filtering theory, which consists of the measurement and time updates, is proposed to give the optimal state estimate for the linear stochastic dynamical systems [1]–[3].

The idea of the Kalman filtering theory is then extended to nonlinear dynamical systems since most of engineering problems are nonlinear in nature, see for examples [4]–[6]. In implementation of the extended Kalman filter (EKF), the Jacobian matrices, which are derived on the state and the measurement output equations, are evaluated with the current predicted states. This linearization would not give the optimal state estimate and the divergence could be happened towards the wrong estimated solution [7]–[8].

Manuscript received Jan 13, 2017

Sie Long Kek, Center for Research on Computational Mathematics, Universiti Tun Hussein Onn Malaysia, Batu Pahat, Johor, Malaysia. (e-mail: slkek@uthm.edu.my).

Kim Gaik Tay, Department of Communication Engineering, Universiti Tun Hussein Onn Malaysia, Batu Pahat, Johor, Malaysia. (e-mail: tay@uthm.edu.my).

Sy Yi Sim, Department of Electrical Engineering Technology, Universiti Tun Hussein Onn Malaysia, Batu Pahat, Johor, Malaysia. (e-mail: sysim@uthm.edu.my).

To improve the EKF, the unscented Kalman filter (UKF) is investigated [9]. In such study, the probability density function is approximated by a deterministic sampling of points using the unscented transformation [10]–[11]. The UKF is more robust and more accurate than the EKF for the estimation errors. However, the UKF does not perform well for the bad initial state and its robustness is less than the optimization based state estimators, for instance, the moving horizon estimator [12]. Practically, state estimation with the Kalman filtering theory has been widely applied in engineering and sciences, which covers target tracking [13], robotic manipulators [14], reservoir modeling [15], biomedical applications [16], sensor data [17], and control systems [18]–[19].

In this paper, we propose an adjustable state estimator, which is based on the association of the Kalman filtering theory and the principle of model-reality differences, for nonlinear estimation problem of discrete-time stochastic dynamical system. In our approach, the adjusted parameters are introduced into the linear dynamical system, both for state and output equations. During the computation procedures, the output from the real plant is measured, in turn, updates the model trajectory iteratively. In this way, the differences between the real plant and the model used are calculated in each iteration step. Consequently, when the convergence is achieved, the optimal solution of the model used approaches to the true optimal solution of the original estimation problem, in spite of model-reality differences. On this basis, an iterative algorithm of the adjustable state estimator is then established for the estimation problem of discrete-time nonlinear stochastic dynamical system.

The rest of the paper is organized as follows. In Section 2, the estimation problem of a general discrete time nonlinear stochastic dynamical system is described. For simplicity, a linear model-based estimation problem, which is added with the adjustable parameters, is formulated. In Section 3, an expanded optimal estimation problem, which takes into account state estimation and parameter estimation, is introduced. The resulting iterative algorithm that is based on the Kalman filtering theory and the principle of model-reality differences is then derived for solving the nonlinear estimation problem. In Section 4, the convergence analysis of the resulting algorithm is given. In Section 5, three illustrative examples are studied for the efficiency. Finally, some concluding remarks are made.

II. Problem Statement

Consider a dynamical system that is governed by the general difference equations given below.

$$x(k+1) = f(x(k),k) + G\omega(k) \tag{1a}$$
$$y(k) = h(x(k),k) + \eta(k) \tag{1b}$$

where $x(k) \in \Re^n$, $k = 0,1, \ ,N$, and $y(k) \in \Re^p$, $k = 0,1, \ ,N$, are, respectively, the state sequence and the output sequence, while $\omega(k) \in \Re^q$, $k = 0,1, \ ,N-1$, and $\eta(k) \in \Re^p$, $k = 0,1, \ ,N$, are the stationary Gaussian white noise sequences with zero mean and their covariance matrices are, respectively, given by Q_ω and R_η . Here, $Q_\omega \in \Re^{q \times q}$ and $R_\eta \in \Re^{p \times p}$ are positive definite matrices, and $G \in \Re^{n \times q}$ is the process noise coefficient matrix. Moreover, $f : \Re^n \times \Re \to \Re^n$ represents the plant dynamics and $h : \Re^n \times \Re \to \Re^p$ is the output measurement channel.

The initial state is

$$x(0) = x_0$$

where $x_0 \in \Re^n$ is a random vector with mean and covariance are, respectively, given by

$$E[x_0] = \bar{x}_0 \text{ and } E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^{\mathrm{T}}] = M_0 .$$

Here, $M_0 \in \Re^{n \times n}$ is a positive definite matrix. It is assumed that initial state, process noise and measurement noise are statistically independent.

Suppose the state mean propagation and the corresponding output measurement are given below.

$$\bar{x}(k+1) = f(\bar{x}(k),k) , \ \bar{x}(0) = \bar{x}_0 \tag{2a}$$
$$\bar{y}(k) = h(\bar{x}(k),k) \tag{2b}$$

where $\bar{x}(k) \in \Re^n$, $k = 0,1, \ ,N$, is the expected state sequence with the state error covariance $M_x(k) \in \Re^{n \times n}$, and $\bar{y}(k) \in \Re^p$, $k = 0,1, \ ,N$, is the expected output sequence with the output error covariance $M_y(k) \in \Re^{p \times p}$. Then, an optimal sequence of the state estimate, denoted by $\hat{x}(k) \in \Re^n$, $k = 0,1, \ ,N-1$, shall be determined over the dynamical system (2) such that the following weighted mean-square error (MSE) is minimized:

$$\begin{aligned} \min_{x(k)} J_{mse}(x) &= \sum_{k=0}^{N} \tfrac{1}{2}(x(k) - \bar{x}(k))^{\mathrm{T}} (M_x(k))^{-1} (x(k) - \bar{x}(k)) \\ &\quad + \tfrac{1}{2}(y(k) - \bar{y}(k))^{\mathrm{T}} (M_y(k))^{-1} (y(k) - \bar{y}(k)) \end{aligned} \tag{3}$$

It is assumed that all functions in (1a), (1b), (2a), (2b) and (3) are continuously differentiable with respect to their respective arguments.

This problem is regarded as a nonlinear estimation problem and is referred to as Problem (P).

In the presence of the random disturbances, the entire trajectory of the state dynamics and the output behavior in Problem (P), which are given by (1a) and (1b), cannot be obtained accurately. Due to the complex structure of Problem (P), a simplified linear dynamics model is formulated from (2) given by

$$\bar{x}(k+1) = A\bar{x}(k) + \alpha_1(k) , \ \bar{x}(0) = \bar{x}_0 \tag{4a}$$
$$\bar{y}(k) = C\bar{x}(k) + \alpha_2(k) \tag{4b}$$

such that the weighted MSE in (3) is minimized, where $\alpha_1(k) \in \Re^n$, $k = 0,1, \ ,N-1$, and $\alpha_2(k) \in \Re^p$, $k = 0,1, \ ,N$, are introduced as the adjustable parameters. Here, $A \in \Re^{n \times n}$ and $C \in \Re^{p \times n}$ are, respectively, the state transition matrix and the output coefficient matrix, where they can be obtained from the linearization of the plant dynamics and the output measurement channel, respectively, at the known initial state.

This problem is a linear estimation problem and is referred to as Problem (M).

Notice that by solving Problem (M) iteratively, we can obtain the true optimal state estimate of Problem (P). This way can be done because of taking into account the differences between Problem (P) and Problem (M) by the adjusted parameters. On the other hand, the output that is measured from the original problem is fed back into the model in order updates the optimal solution of the model repeatedly. Follow from this updating step, the optimal solution of Problem (M) approximates to the true optimal solution of Problem (P), in spite of model-reality differences, once the convergence is achieved.

III. Integrated Approach with Model-Reality Differences

Now, let us introduce an expanded optimal estimation problem, which is referred to as Problem (E), given below.

$$\begin{aligned} \min_{x(k)} J_{mse}(x) &= \sum_{k=0}^{N} \tfrac{1}{2}(x(k) - \bar{x}(k))^{\mathrm{T}} (M_x(k))^{-1} (x(k) - \bar{x}(k)) \\ &\quad + \tfrac{1}{2}(y(k) - \bar{y}(k))^{\mathrm{T}} (M_y(k))^{-1} (y(k) - \bar{y}(k)) \\ &\quad + \tfrac{1}{2} r_1 \ \bar{x}(k) - z(k) \ ^2 \end{aligned}$$

subject to (5)

$$\bar{x}(k+1) = A\bar{x}(k) + \alpha_1(k) , \ \bar{x}(0) = \bar{x}_0$$
$$\bar{y}(k) = C\bar{x}(k) + \alpha_2(k)$$
$$Az(k) + \alpha_1(k) = f(z(k),k)$$
$$Cz(k) + \alpha_2(k) = h(z(k),k)$$
$$z(k) = \bar{x}(k)$$

where $z(k) \in \Re^n$, $k = 0,1, \ ,N$, is introduced to separate the optimal state sequence in the state estimation problem from the respective signal in the parameter estimation, and

$\|\cdot\|$ denotes a usual Euclidean norm. The term of $\frac{1}{2}r_1\|\bar{x}(k)-z(k)\|^2$ with $r_1 \in \Re$ is introduced to improve the convexity and to facilitate the convergence of the resulting iterative algorithm. It is important to note that the algorithm is designed such that $z(k)=\bar{x}(k)$ is satisfied upon termination of the iterations, assuming that the convergence is achieved. The state estimate $z(k)$ is used for the computation of parameter estimation and the matching scheme, while the corresponding state estimate $\bar{x}(k)$ will give the optimal state sequence for state estimation. Thus, state estimation and parameter estimation are mutually interactive.

A. *State estimation*

By taking the first-order necessary condition $dJ_{mse}(x)=0$ in (5) for arbitrary $dx(k)$, the coefficients must vanish. After carrying out some algebraic manipulations, the optimal state sequence is yielded as follows:

$$\hat{x}(k)=\bar{x}(k)+K_f(k)(y(k)-\bar{y}(k)) \qquad (6)$$

where

$$K_f(k)=M_x(k)C^{\mathrm{T}}M_y(k)^{-1} \qquad (7)$$

is the filter gain matrix. This state information improves the expected state sequence and the corresponding measured output sequence given in (4a) and (4b), where the deterministic dynamic system is propagated to generate the following optimal state sequence and the corresponding measured output sequence,

$$\bar{x}(k+1)=A\bar{x}(k)+K_p(k)(y(k)-\bar{y}(k))+\alpha_1(k) \qquad (8a)$$

$$\bar{x}(0)=\bar{x}_0$$

$$\bar{y}(k)=C\bar{x}(k)+\alpha_2(k) \qquad (8b)$$

Here, $K_p(k)\in\Re^{n\times p}$ is the predictor gain matrix whose computation is given by

$$K_p(k)=AM_x(k)C^{\mathrm{T}}M_y(k)^{-1} \qquad (9a)$$

$$M_y(k)=CM_x(k)C^{\mathrm{T}}+R_\eta \qquad (9b)$$

$$M_x(k+1)=AM_x(k)A^{\mathrm{T}}-K_p(k)M_y(k)K_p(k)^{\mathrm{T}}+GQ_\omega G^{\mathrm{T}},\ M_x(0)=M_0 \qquad (9c)$$

where $M_y(k)\in\Re^{p\times p}$ and $M_x(k)\in\Re^{n\times n}$ are positive definite matrices [7]–[8], [20]–[21].

Remarks:

(a) The filter gain $K_f(k)$, which minimizes the weighted MSE of the state estimation, and the predictor gain $K_p(k)$ converge asymptotically to a unique constant gain given, respectively, by K_f and K_p as k tends to ∞.

(b) The state error covariance $M_x(k)$ and the output error covariance $M_y(k)$ converge asymptotically to a unique constant error covariance given, respectively, by M_x and M_y as k tends to ∞.

B. *Parameter estimation*

The separable variable

$$z(k)=\bar{x}(k)$$

uses the estimated parameter to match the nonlinear dynamics during each iteration step. On this basis, the differences between the plant dynamics, including the output behavior, in Problem (P) and the model used in Problem (M) can be calculated repeatedly by

$$\alpha_1(k)=f(z(k),k)-Az(k) \qquad (10a)$$

$$\alpha_2(k)=h(z(k),k)-Cz(k) \qquad (10b)$$

It is noticed that the model used is updated repetitively so as the converged solution approaches to the true optimal solution of Problem (P), in spite of model-reality differences.

C. *Iterative algorithm*

From the discussion above, state estimation and parameter estimation are integrated for solving the nonlinear estimation problem of the stochastic dynamical system based on the proposed linear estimation model. Here, we summarize the solution method as an iterative algorithm given below.

Algorithm 1: Iterative algorithm

Data A, C, G, M_0, N, Q_ω, R_η, $\bar{x}_0$, r_1, k_z, f, h. Note that A and C might be obtained from the linearization of f and h, respectively.

Step 1 Compute $K_p(k)$, $M_x(k)$ and $M_y(k)$ from (9a), (9b) and (9c), respectively.

Step 2 Assume $\alpha_1(k)=0$ and $\alpha_2(k)=0$. Calculate the state estimate sequence $\bar{x}(k)$ and the corresponding output sequence $\bar{y}(k)$ from (8a) and (8b) with the given initial condition $\bar{x}(0)=\bar{x}_0$. Set $i=0$, $z(k)^0=\bar{x}(k)^0$.

Step 3 Compute the nonlinear dynamics of $f(z(k)^i,k)$ and $h(z(k)^i,k)$ from (2a) and (2b).

Step 4 Compute the adjustable parameters $\alpha_1(k)^i$ and $\alpha_2(k)^i$ from (10a) and (10b). This step is called the *parameter estimation* step.

Step 5 With the specific $\alpha_1(k)^i$ and $\alpha_2(k)^i$, compute the new $\bar{x}(k)^i$ and the new $\bar{y}(k)^i$ from (8a) and (8b). This step is called the *state estimation* step.

Step 6 Test convergence of the optimal sequence of the state estimate for Problem (P). To provide a mechanism for regulating convergence, a simple relaxation method is employed as given below:

$$z(k)^{i+1}=z(k)^i+k_z(\bar{x}(k)^i-z(k)^i) \qquad (11)$$

where $k_z \in (0,1]$ is a scalar gain. If $z(k)^{i+1} = z(k)^i$ within a given tolerance ε, stop; else set $i = i+1$ and repeat from Step 3.

Remarks:

(a) $K_p(k)$, $M_x(k)$ and $M_y(k)$ are computed off-line in Step 1 for the optimal sequences of the state estimate and the measured output. Their values would not change during the iterations.

(b) The convergence of $z(k)$ in Step 6 is verified by comparing the following 2-norm with the given tolerance

$$\| z^{i+1} - z^i \|_2 = \sqrt{\sum_{k=0}^{N} \| z(k)^{i+1} - z(k)^i \|^2} \tag{12}$$

(c) The value of scalar gain k_z is set to 0.9 as a default value, and this value can be chosen from 0.1 to 0.9 for an optimal number of iteration.

IV. Convergence Analysis

In this section, our main aim is to show that, under the principle of model-reality differences, the adjustable state estimator can identify the nonlinear dynamics of stochastic system.

To obtain our main result, the following assumption is needed:

Assumption 1

(a) The state estimate sequence $\bar{x}^*(k)$ and the corresponding measured output sequence $\bar{y}^*(k)$ of Problem (P) exist and they are unique.

(b) The functions f and h are continuously differentiable.

In addition to this, we have the following theorem:

Theorem 1

Under Assumption 1 and assuming the convergence is achieved, the converged state estimate $\bar{x}^c(k)$ and the corresponding measured output $\bar{y}^c(k)$ of Problem (M) are the optimal solution of Problem (P).

Proof

Consider the necessary condition, that is $dJ_{mse}(x) = 0$, the nonlinear state estimate and the output measurement are given by

$$\bar{x}^*(k+1) = f(\bar{x}^*(k), k) + K_p(k)(y(k) - \bar{y}^*(k)),$$

$$\bar{x}^*(0) = \bar{x}_0 \tag{13a}$$

$$\bar{y}^*(k) = h(\bar{x}^*(k), k). \tag{13b}$$

Meanwhile, the converged optimal state estimate and the converged output measurement are given as follow:

$$\bar{x}^c(k+1) = A\bar{x}^c(k) + K_p(k)(y(k) - \bar{y}^c(k)) + \alpha_1(k),$$

$$\bar{x}^c(0) = \bar{x}_0 \tag{14a}$$

$$\bar{y}^c(k) = C\bar{x}^c(k) + \alpha_2(k) \tag{14b}$$

where

$$\alpha_1(k) = f(z(k), k) - Az(k) \tag{15a}$$

$$\alpha_2(k) = h(z(k), k) - Cz(k) \tag{15b}$$

This implies that the following results are valid:

$$\bar{x}^c(k+1) = f(\bar{x}^c(k), k) + K_p(k)(y(k) - \bar{y}^c(k)),$$

$$\bar{x}^c(0) = \bar{x}_0 \tag{16a}$$

$$\bar{y}^c(k) = h(\bar{x}^c(k), k) \tag{16b}$$

where $z(k) = \bar{x}^c(k)$ is satisfied as the convergence is achieved.

Comparing (16a) and (16b) with (13a) and (13b), we can conclude that these state estimate and output measurement are equivalent. Thus, we obtain

$$\bar{x}^c(k) = \bar{x}^*(k) \quad \text{and} \quad \bar{y}^c(k) = \bar{y}^*(k).$$

This completes the proof. ♦

V. Illustrative Examples

For illustration, three examples are studied. Their simplified models, which are added with the adjustable parameters, are constructed for solving the original estimation problems. The nonlinearity of each example is different. The plant dynamics of the first and the third examples are rational terms with one-dimensional sinusoidal for the first example, and two-dimensional for the third example, while the plant dynamics of the second example is a two-dimensional bilinear form. The output measurement of the first example is quadratic term, and the output measurements are linear form for the second and the third examples. Their simulation and graphical results are given afterward.

A. Example 1

Consider a nonlinear dynamical system that is disturbed by the stationary Gaussian white noise sequences [22]–[23] given below:

$$x(k+1) = 0.5x(k) + 25x(k)(1 + (x(k))^2)^{-1} + 8\cos(1.2k) + \omega(k)$$

$$y(k) = 0.05(x(k))^2 + \eta(k)$$

where the initial state $x(0) = x_0$ is a random vector with mean of 0.1 and unit variance, $\omega(k)$ and $\eta(k)$ are Gaussian white noise sequences with zero mean and unit variance. This problem is referred to as Problem (P), while its simplified model, which is referred to as Problem (M), is given below:

$$\bar{x}(k+1) = 0.5\bar{x}(k) + \alpha_1(k)$$
$$\bar{y}(k) = 0.01\bar{x}(k) + \alpha_2(k)$$

with the initial condition $\bar{x}(0) = 0.1$, and the adjustable parameters $\alpha_1(k)$ and $\alpha_2(k)$, for $k = 0,1,...,50$.

The simulation result is shown in Table 1, where there is an 83 percent of the error reduction after performing the algorithm proposed. The final MSE, which is 0.3870, is less than the MSE of the EKF, which is 1.3701. The nonlinear dynamics of the plant and the state estimate are shown in Fig. 1, and the output sequence both for the original output and the model output are shown in Fig. 2. The trajectories of state and output are fluctuated seasonally because of the attending of the cosine term in the real plant, where the values of the output measurement are always positive.

Table 1. Simulation result, Example 1

Number of iteration	Elapsed time	Initial MSE	Final MSE
55	0.235294	2.3300	0.3870

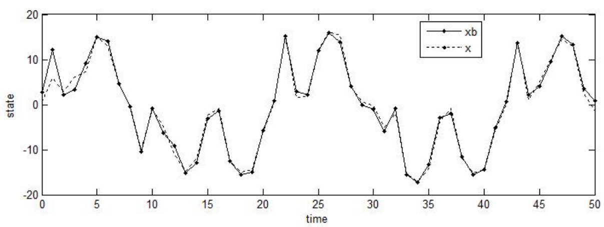


Fig. 1. State trajectories, Example 1

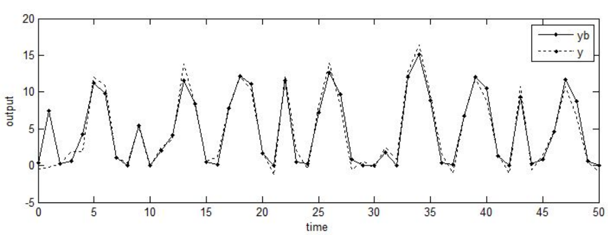


Fig. 2. Output trajectories, Example 1

B. Example 2

Consider the following nonlinear dynamical system [24]–[25]:

$$x_1(k+1) = 0.8x_1(k) + x_1(k)x_2(k) + 0.1 + 0.01\omega_1(k)$$
$$x_2(k+1) = 1.5x_2(k) - x_1(k)x_2(k) + 0.1 + 0.01\omega_2(k)$$
$$y(k) = x_2(k) + 0.01\eta(k)$$

where the initial state $x(0) = x_0$ has a mean vector of $\bar{x}_0 = (1.35, 0.11)^{\mathrm{T}}$ and a covariance matrix of $M_0 = I_2$, $\omega(k) = (\omega_1(k), \omega_2(k))^{\mathrm{T}}$ and $\eta(k)$ are zero mean Gaussian white noise sequences with their respective covariance matrices given by $Q_\omega = 10^{-4} I_2$ and $R_\eta = 10^{-4}$. Let us define this problem as Problem (P) and the simplified model as Problem (M) which is given below:

$$\bar{x}_1(k+1) = 0.91\bar{x}_1(k) + 1.35\bar{x}_2(k) + \alpha_{11}(k)$$
$$\bar{x}_2(k+1) = -0.11\bar{x}_1(k) + 0.15\bar{x}_2(k) + \alpha_{12}(k)$$
$$\bar{y}(k) = \bar{x}_2(k) + \alpha_2(k)$$

with the initial condition

$$\bar{x}_1(0) = 1.35, \quad \bar{x}_2(0) = 0.11, \quad k = 0,1,...,80$$

and the adjusted parameters $\alpha_1(k) = (\alpha_{11}(k) \quad \alpha_{12}(k))^{\mathrm{T}}$ and $\alpha_2(k)$.

In Table 2, the simulation result shows that there is a 98 percent of the error reduction after running the algorithm proposed. The final MSE of 0.0028 is superior to the MSE of the EKF that is 0.0041. The trajectories for the plant dynamics and the state estimate are shown in Fig. 3, where the state estimate tracks the plant dynamics closely. The trajectories for the original output and the model output, where the variation occurs around 0.12, are shown in Fig. 4.

Table 2. Simulation result, Example 2

Number of iteration	Elapsed time	Initial MSE	Final MSE
78	0.116620	0.1282	0.0028

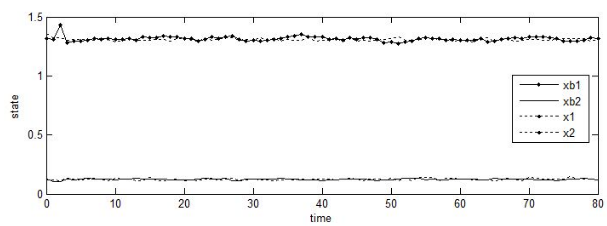


Fig. 3. State trajectories, Example 2

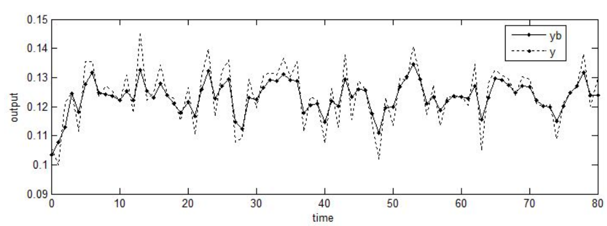


Fig. 4. Output trajectories, Example 2

C. Example 3

Consider a nonlinear dynamical system [26] in Problem (P) given below:

$$x_1(k+1) = 0.99x_1(k) + 0.2x_2(k)$$

$$x_2(k+1) = -0.1x_1(k) + \frac{0.5x_2(k)}{1+(x_2(k))^2} + \omega(k)$$
$$y(k) = x_1(k) - 3x_2(k) + \eta(k)$$

where the initial condition $x(0) = x_0$ is a random vector with mean and covariance are, respectively, given by

$$\bar{x}_0 = \begin{bmatrix} 1.0 \\ 0.8 \end{bmatrix} \text{ and } M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The stationary Gaussian white noise sequences are $\omega(k)$ and $\eta(k)$ with zero mean and their respective covariance matrices are given by

$$Q_\omega = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } R_\eta = 1 .$$

The simplified model in Problem (M) is given below:

$$\bar{x}_1(k+1) = 0.99\bar{x}_1(k) + 0.2\bar{x}_2(k) + \alpha_{11}(k)$$
$$\bar{x}_2(k+1) = -0.1\bar{x}_1(k) + 0.95\bar{x}_2(k) + \alpha_{12}(k)$$
$$\bar{y}(k) = \bar{x}_1(k) - 3\bar{x}_2(k) + \alpha_2(k)$$

with the initial condition

$$\bar{x}_1(0) = 1.0\,,\quad \bar{x}_2(0) = 0.8\,,\quad k = 0,1,...,20$$

and the adjusted parameters $\alpha_1(k) = (\alpha_{11}(k) \quad \alpha_{12}(k))^{\mathrm{T}}$ and $\alpha_2(k)$.

Table 3 shows the simulation result, where there is a 46 percent of the error reduction done by the algorithm proposed. Furthermore, the final MSE is preferred since the value of 0.2863 is smaller than the MSE of the EKF with 0.4468. The dynamics of the plant and state estimate are shown in Fig. 5, where the state estimate tracks the plant dynamics slightly. The output behaviors for the original output and the model output, which are similar equivalently, are shown in Fig. 6.

Table 3. Simulation result, Example 3

Number of iteration	Elapsed time	Initial MSE	Final MSE
37	0.032799	0.5293	0.2863

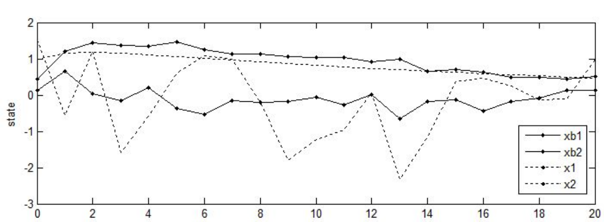


Fig. 5. State trajectories, Example 3

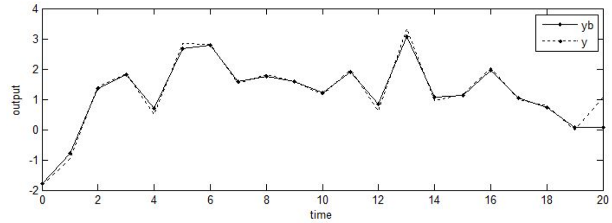


Fig. 6. Output trajectories, Example 3

VI. Concluding Remarks

In this paper, an adjustable state estimator was discussed for solving the nonlinear estimation problem of the discrete-time stochastic dynamical systems. By introducing the adjusted parameters into the simplified linear model-based estimation problem, the differences between the real plant and the model used could be taken into account during the computation procedure. In this way, the real output is measured and then it is fed back into the model used. Therefore, the trajectories of model and output of the linear model-based estimation problem are updated repeatedly. Consequently, the iterative solution converges to the true optimal solution of the original estimation problem despite model-reality differences when the convergence is achieved. For illustration, three examples were studied and the results show the efficiency of the algorithm proposed. In conclusion, the applicable of the algorithm proposed to nonlinear estimation problem is highly recommended.

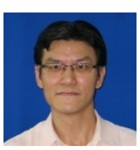
Acknowledgment

The authors would like to thank the Universiti Tun Hussein Onn Malaysia (UTHM) for financial supporting to this study under Incentive Grant Scheme for Publication (IGSP) VOT. U417.

References

[1] R. E. Kalman, “A new approach to linear filtering and prediction problems”, *Journal of Basic Engineering*, 35-45, 1960a.
[2] R. E. Kalman, “Contributions to the theory of optimal control”, *Bol. Soc. Mat. Mexicana*, 102-119, 1960b.
[3] R. E. Kalman and R.S. Bucy, “New results in linear filtering and prediction theory”, *Journal of Basic Engineering*, 95-108, 1961.
[4] B. A. McElhoe, “An assessment of the navigation and course corrections for a manned flyby of Mars or Venus”, *Aerospace and Electronic Systems, IEEE Transactions on*, **AES-2**, 613-623, 1966.
[5] G. L. Smith, S. F. Schmidt and L. A. McGee, *Application of statistical filter theory to the optimal estimation of position and velocity on board a circumlunar vehicle*, United States: National Aeronautics and Space Administration, 1962.
[6] N. U. Ahmed, *Linear and Nonlinear Filtering for Scientists and Engineers*. World Scientific Publishers, Singapore, New Jersey, London, Hong Kong, 1999.
[7] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs NJ: Prentice-Hall, 1979.
[8] A. Bagchi, *Optimal Control of Stochastic Systems*, New York: Prentice-Hall, 1993.
[9] S. J. Julier and J. Uhlmann, “Unscented filtering and nonlinear estimation”, *Proceedings of the IEEE*, **92**, 401–422, 2004.
[10] J. Uhlmann, *Dynamic Map Building and Localization: New Theoretical Foundations*, Ph.D. thesis, University of Oxford, 1995.
[11] S. Julier and J. Uhlmann, “A new method for the nonlinear transformation of means and covariances in nonlinear filters”, *IEEE Trans. on Automatic Control*, 45, 477-482, 2000.

[12] M. R. Rajamani, *Data-Based Techniques to Improve State Estimation in Model Predictive Control*, Ph.D. thesis, University of Wisconsin-Madison, 2007.
[13] S. R. Olfati, "Collaborative target tracking using distributed Kalman filtering on mobile sensor networks", *American Control Conference* (ACC), 29 June-1 July, Dartmouth Coll., Hanover, NH, USA, 1100-1105, 2011.
[14] G. G. Rigatos, "Derivative-free nonlinear Kalman filtering for MIMO dynamical systems: application to multi-DOF robotic manipulators", *International Journal Advanced Robotic Systems*, **8** (6), 47-61, 2011.
[15] M. V. Krymskaya, R. G. Hanea and M. Verlaan, "An iterative ensemble Kalman filter for reservoir engineering applications", *Computational Geosciences*, **13** (2), 235-244, 2009.
[16] F. Andrea, S. Giovanni and C. Claudio, "Enhanced accuracy of continuous glucose monitoring by online extended Kalman filtering", *Diabetes Technology & Therapeutics*, **12** (5), 353-363, 2010.
[17] Z. G. Feng, K. L. Teo, N. U. Ahmed, Y. Zhao and W. Y. Yan, "Optimal fusion of sensor data for Kalman filtering", *Discrete and Continuous Dynamical Systems–Series A*, **14** (3), 483-503, 2006.
[18] S. L. Kek, K. L. Teo and A. A. Mohd Ismail, "An integrated optimal control algorithm for discrete-time nonlinear stochastic system", *International Journal of Control*, **83** (12), 2536-2545, 2010.
[19] S. L. Kek, K. L. Teo and A. A. Mohd Ismail, "Filtering solution of nonlinear stochastic optimal control problem in discrete-time with model-reality differences", *Numerical Algebra, Control and Optimization* (NACO), **2** (1), 207-222, 2012.
[20] A. E. Bryson and Y. C. Ho, *Applied Optimal Control*, Washington, DC: Hemisphere, 1975.
[21] Y. Bar-Shalom, X. R. Li and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*. New York: John Wiley & Sons, Inc, 2001.
[22] D. Simon, *Optimal State Estimation: Kalman, H-infinity and Nonlinear Approaches*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.
[23] M. L. Andrade Netto, L. Gimeno and M. J. Mendes, "On the optimal and suboptimal nonlinear filtering problem for discrete-time systems", *IEEE Trans. Automat. Control* **23**, 1062-1067, 1978.
[24] A. Germani, C. Manes and P. Palumbo, "Polynomial extended Kalman filtering for discrete-time nonlinear stochastic systems", *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA, 886-891, 2003.
[25] X. Sun, L. Jin and M. Xiong, "Extended Kalman filter for estimation of parameters in nonlinear state-space models of biochemical networks", PLoS ONE 3 (11): e3758.doi:10.1371/journal.pone.0003758, 2008.
[26] Y. Wu, D. Hu, M. Wu and X. Hu, "Unscented Kalman filtering for additive noise case: augmented vs. non-augmented", *American Control Conference*, June 8-10, Portland, OR, USA, 4051-4055, 2005.



Sie Long Kek received the M.Sc. degree and the Ph.D. in mathematics from the Universiti Teknologi Malaysia, Johor, Malaysia, in 2002 and 2011, respectively. He is currently a Senior Lecturer in the Department of Mathematics and Statistics, Universiti Tun Hussein Onn Malaysia. His research interests includes optimization and control, operation research and management science, and modelling and simulation.

Kim Gaik Tay received the M.Sc. degree and the Ph.D. degree in mathematics from the Universiti Teknologi Malaysia, Johor, Malaysia, in 2000 and 2007, respectively. She is currently an Associate Professor in the Department of Communication Engineering, Universiti Tun Hussein Onn Malaysia. Her research interests includes nonlinear waves, optical solitons and numerical methods.

Sy Yi Sim received the Bachelor degree (first class Honours) and Ph.D. degree in electrical engineering from the Universiti Tun Hussein Onn Malaysia, Johor, Malaysia, in 2012 and 2016, respectively. She is currently a Lecturer in the Department of Electrical Engineering Technology, Universiti Tun Hussein Onn Malaysia. Her research interests include energy and green technology, power electronic and renewable energy.