Exponentiated Gumbel Shape Parameter Bayesian Estimation [Singly Type II Samples]

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Abstract— **In this search: The Bayesian analysis of the shape parameter Exponentiated Gumbel distribution has been considered.The estimation has been obtained under[weighed and linear] loss functions for two different prior distribution [Hypothetical and Exponential].The estimation made under single type II data analyses. The Simulation study has been conducted to compare by mean squared error [MSE] for the performance of the four different estimators by nine assuming experiments for different Values of distribution parameters and different sample sizes**

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I. INTRODUCTION

The Gumbel distribution is named by Emil Julius Gumbel (1891-1966) , based on his original papers describing the distribution[6] .The Gumbel distribution has been introduced originally for Modeling failure data , it is called extreme value type I distribution obtained by Emil Julius Gumbel (1935) [5] .This distribution might be used to represent the distribution of the maximum Level of ariver in a particular year if there was a List of maximum values for the past ten years . it is useful in predicting chance that an extrem earthquake , flood or other natural disaster will occur[1]

Here used the Bayesian analysis to estimate the shape parameter(α) for Exponentiated Gumbel distribution[7] .
The commutative distribution function for [EGD] is given by :[2]

$$F(x) = [\exp\{-\exp(-\lambda x)\}]^{\alpha}$$

Then the density function as:

$$f(x,\lambda,\alpha) = \lambda\alpha[-exp(-\lambda x)\}]^{\alpha} e^{-\lambda x}$$

are the scale and shape parameter respectively. Where (λ,α)

The concept of the research requires using of statistical distribution as a model of failure used Bayes method to estimate the shape parameter assumming that the scale parameter is known (λ=1) with two informative and non -informative prior functions under two symmetric and asymmetric Loss functions using simulation study to obtain the best estimator for this shape parameter then the pdf become:

……..(1) $f(x,\alpha) = \alpha[\exp\{-exp(-x)\}]^{\alpha} e^{-x}$

and the commutative distribution function as:

$$F(x,\alpha) = [\exp\{-exp(-x)\}]^{\alpha} \quad \text{.........(2)}$$

II. AIM OF RESEARCH

The aim of research is to estimate the shape parameter of Exponentiated Gumbel distribution using Bayesian estimation , then find the best by comparison among estimators by MSE using simulation study.

III. BAYESIAN ESTIMATION

Let x_1,x_2, …..,x_n iid. , f(x,α) , the Bayesian views the density of each data observation as a conditional density , we can update our beliefs about the parameter α by computing the posterior density [3]:

$$p(\alpha|\underline{x}) = \frac{L(\underline{x}|\alpha)\, p(\alpha)}{\int_0^{\infty} L(\underline{x}|\alpha)\, p(\alpha)\, d\alpha} \quad \text{.........(3)}$$

Where : L(x|α) is the likelihood for the density function.
p(α) is the prior function for the shape parameter.

IV. LOSS FUNCTION

A loss function L($\hat{\alpha},\alpha$) represents losses incurred when we estimate the parameter α by $\hat{\alpha}$.[9]

Weighted loss Function 4-1

This is from symmetric type given by $L(\hat{\alpha},\alpha) = \frac{(\hat{\alpha}-\alpha)}{\alpha}$, and the Bayes estimator for the parameter is :

$$\hat{\alpha}_W = \frac{1}{E(\alpha^{-1}|\underline{x})} \quad \text{.........(4)}$$

4-2 Linear exponential loss Function

This is from a symmetric type given by:
$L(\hat{\alpha},\alpha) = (e^{c\Delta i} - c\Delta i - 1)$, and the Bayes estimator for the parameter is :

$$\hat{\alpha}_L = -\frac{1}{c}\ln E(e^{-c\alpha}|\underline{x}) \quad \text{.........(5)}$$

V. BAYESIAN ANALYSES

Substitute the equation (1) ,
The likelihood function $L(\alpha|\underline{x}) = \prod_{i=1}^{n} f(xi;\alpha)$ is given by:

$$L(\alpha|\underline{x}) = \frac{n!}{(n-r)!}\alpha^{r} e^{\sum_{i=1}^{n} x_i} e^{-\alpha\sum_{i=1}^{n} e^{-x_i}} \sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{j} e^{-\alpha j e^{-x_i}}$$

$$\therefore L(\alpha|\underline{x}) = \frac{n!}{(n-r)!}\alpha^r \sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j e^{-\sum_{i=1}^{n} x_i} e^{-\alpha\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)} \quad \ldots\ldots(6)$$

The posterior distribution under Hypothetical prior. 5-1

The prior function given by :

$g(\alpha) = k\alpha^3$ $\quad k, \alpha > 0$ ……(7) To find the posterior under this prior , substitute the equations (6) &(7) in (3) , will be as :

$$p(\alpha|\underline{x}) = \frac{\frac{n!}{(n-r)!}k\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j e^{-\sum_{i=1}^{n} x_i}\alpha^{r+3} e^{-\alpha\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)}}{\frac{n!}{(n-r)!}k\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j e^{-\sum_{i=1}^{n} x_i}\int_0^\infty \alpha^{r+3} e^{-\alpha\left(\sum_{i=0}^{n} e^{-x_i}+je^{-x_r}\right)}\,d\alpha}$$

Then the posterior function under Hypothetical prior is

$$p(\alpha|\underline{x}) = \frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\alpha^{r+3} e^{-\alpha\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)}}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\frac{\Gamma r+4}{\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)^{r+4}}} \quad \ldots\ldots\ldots(8)$$

Bayes estimators 1 5-1-

The Bayes estimator for the shape parameter under Hypothetical prior use the two loss functions is:

1) Weighted loss function

The Bayes estimator use this loss function from equation (4) and (8) is given by:

$$\hat{\alpha}_{WH} = \frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j(r+3)}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)} \quad \ldots\ldots\ldots(9)$$

2) Linear exponential loss function

The Bayes estimator use this loss function from equation (5) and (8) is given by:

$$\hat{\alpha}_{LH} = -\frac{1}{c}\ln\left[\frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j}\left(\frac{\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}}{c+\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)}\right)^{r+4}\right] \quad \ldots\ldots\ldots(10)$$

The posterior function under Exponential prior 5-2

The exponential prior is define as :[10]

$g(\alpha) = \theta e^{-\theta\alpha}$ $\quad \theta, \alpha > 0$ ………(11)

To find the posterior distribution under this prior , substitute the equations (6) & (11) in (3), will be as:

$$p(\alpha|\underline{x}) = \frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\alpha^r e^{-\alpha\left[\theta+\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)\right]}}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\frac{\Gamma r+1}{\left[\theta+\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)\right]^{r+1}}} \quad \ldots\ldots(12)$$

Bayes estimators 5-2-1

the Bayes estimator for the shape parameter under Exponential prior use the two loss functions is:

1) Weighted loss function

The Bayes estimator use this loss function from equation (4) and (12) is given by:

$$\hat{\alpha}_{WE} = \frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j(r)}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[\theta+\left(\sum_{i=1}^{n} e^{-x_i}+je^{-x_r}\right)\right]} \quad \ldots\ldots\ldots(13).$$

Linear exponential loss function 2)

The Bayes estimator use this loss function from equation (5) and (12) is given by:

$$\hat{\alpha}_{LE} = -\frac{1}{c}\ln\left[\frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j}\left(\frac{\theta+(\sum_{i=1}^{n}e^{-x_i}+je^{-x_r})}{[c+(\theta+(\sum_{i=1}^{n}e^{x_i}+je^{-x_r}))]}\right)^{r+1}\right] \quad \ldots\ldots(14)$$

VI. Simulation Procedure

To compare between estimators which is the best to estimate the shape parameter , simulation procedure are used , taking different sample size (n=10,25,50,75,100) in nine experiments , by replicating each experiment (R= 1000) , equation (2) is to generating different values of (x) by :U=F(x) , where U is a random variable on interval (0,1) ,then:

$$X = -\ln\left[-\ln U^{\frac{1}{\alpha}}\right] \quad \ldots\ldots\ldots(15)$$

It is Monte – Carlo ś method [10]to format of data generating(15)according to the values of (r) and by sample sizes in nine experiments by tables (1),(2) below:

Table(1) Sample Values

n	10	25	50	75	100
r	3	6	20	45	60

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Table(2) The default Values of parameter

Experiment	α	θ
1	0.1	0.6
2	0.1	1
3	0.1	1.4
4	0.7	0.6
5	0.7	1
6	0.7	1.4
7	1	0.6
8	1	1
9	1	1.4

Then using MSE to compare the best estimator for the shape parameter as:

$$MSE(\hat{\alpha}) = \frac{1}{R}\sum_{i=1}^{n}(\hat{\alpha}-\alpha)^2 \quad \ldots\ldots\ldots(16)$$

from the estimators in equations (9),(10,(13)and (14).

Conclusions

From table (3,4) belong experiments (1&2) which contains the simulation results of MSE (eq.16) for Bayes estimator the shape parameter α , the best under Exponential prior using Linear loss function in (n = 10,25,50), where in large sample sizes (75&100),the best estimator under the same prior using two loss functions (weighted and linear), but in these two experiments and under the Hypothetical prior the best estimator using weighted loss function in all sample sizes .

From tables (5,6,7,8,9,10&11) belong experiments (3,4,5,6,7,8&9) ,and in general the best estimator of the shape parameter of the Exponentiated Gumbel distribution for single type II censored data is under Exponential prior using Linear loss function ,and under Hypothetical prior using weighted loss function in all sample sizes.

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Table (3) MSE Values for Bayesian estimator
Experiment (1)

n	Exponential Prior		Best
	W_E	L_E	
10	0.0069	0.0046	L_E
25	0.0084	0.0075	L_E
50	0.0089	0.0086	L_E
75	0.0088	0.0088	W_E&L_E
100	0.0091	0.0091	W_E&L_E
	Hypothetical Prior		
	W_H	L_H	
10	0.0045	0.0045	W_H&L_H
25	0.0077	0.0078	W_H
50	0.0087	0.0090	W_H
75	0.0087	0.0092	W_H
100	0.0090	0.0094	W_H

Table (4) MSE Values for Bayesian estimator
Experiment (2)

n	Exponential Prior		Best
	W_E	L_E	
10	0.0070	0.0038	L_E
25	0.0085	0.0070	L_E
50	0.0088	0.0082	L_E
75	0.0088	0.0085	L_E
100	0.0091	0.0089	L_E
	Hypothetical Prior		
	W_H	L_H	
10	0.0046	0.0046	W_H&L_H
25	0.0078	0.0079	W_H
50	0.0087	0.0090	W_H
75	0.0087	0.0092	W_H
100	0.0090	0.0094	W_H

Table (5) MSE Values for Bayesian estimator
Experiment (3)

n	Exponential Prior		Best
	W_E	L_E	
10	0.0069	0.0030	L_E
25	0.0084	0.0063	L_E
50	0.0088	0.0079	L_E
75	0.0088	0.0082	L_E
100	0.0091	0.0087	L_E
	Hypothetical Prior		
	W_H	L_H	

10	0.0045	0.0045	W_H&L_H
25	0.0077	0.0078	W_H
50	0.0087	0.0090	W_H
75	0.0087	0.0092	W_H
100	0.0090	0.0094	W_H

Table (6) MSE Values for Bayesian estimator
Experiment (4)

n	Exponential Prior		Best
	W_E	L_E	
10	0.3397	0.2268	L_E
25	0.4152	0.3698	L_E
50	0.4339	0.4204	L_E
75	0.4317	0.4311	L_E
100	0.4450	0.4453	W_E
	Hypothetical Prior		
	W_H	L_H	
10	0.2178	0.2192	W_H
25	0.3797	0.3859	W_H
50	0.4257	0.4410	W_H
75	0.4279	0.4508	W_H
100	0.4428	0.4608	W_H

Table (7) MSE Values for Bayesian estimator
Experiment (5)

n	Exponential Prior		Best
	W_E	L_E	
10	0.3440	0.1853	L_E
25	0.4141	0.3401	L_E
50	0.4341	0.4042	L_E
75	0.4317	0.4170	L_E
100	0.4451	0.4345	L_E
	Hypothetical Prior		
	W_H	L_H	
10	0.2203	0.2222	W_H
25	0.3778	0.3838	W_H
50	0.4259	0.4411	W_H
75	0.4279	0.4508	W_H
100	0.4428	0.4608	W_H

Table (8) MSE Values for Bayesian estimator
Experiment (6)

n	Exponential Prior		Best
	W_E	L_E	
10	0.3478	0.1535	L_E
25	0.4147	0.3149	L_E
50	0.4338	0.3874	L_E
75	0.4319	0.4033	L_E
100	0.4451	0.4239	L_E
	Hypothetical Prior		
	W_H	L_H	
10	0.2238	0.2256	W_H
25	0.3782	0.3842	W_H
50	0.4255	0.4408	W_H
75	0.4281	0.4509	W_H
100	0.4429	0.4608	W_H

Table (9) MSE Values for Bayesian estimator
Experiment (7)

n	Exponential Prior		Best
	W_E	L_E	
10	0.6981	0.4696	L_E
25	0.8481	0.7558	L_E
50	0.8843	0.8565	L_E
75	0.8810	0.8797	L_E
100	0.9086	0.9093	W_E
	Hypothetical Prior		
	W_H	L_H	
10	0.4464	0.4520	W_H
25	0.7755	0.7884	W_H
50	0.8674	0.8989	W_H
75	0.8732	0.9200	W_H
100	0.9041	0.9406	W_H

Table (10) MSE Values for Bayesian estimator
Experiment (8)

n	Exponential Prior		Best
	W_E	L_E	
10	0.7114	0.3968	L_E
25	0.8480	0.6996	L_E
50	0.8860	0.8250	L_E
75	0.8820	0.8522	L_E
100	0.9085	0.8871	L_E
	Hypothetical Prior		
	W_H	L_H	
10	0.4609	0.4679	W_H
25	0.7742	0.7868	W_H
50	0.8692	0.9003	W_H
75	0.8743	0.9207	W_H
100	0.9039	0.9405	W_H

Table (11) MSE Values for Bayesian estimator
Experiment(9)

n	Exponential Prior		Best
	W_E	L_E	
10	0.7145	0.3165	L_E
25	0.8494	0.6490	L_E
50	0.8847	0.7895	L_E
75	0.8815	0.8232	L_E
100	0.9087	0.8656	L_E
	Hypothetical Prior		
	W_H	L_H	
10	0.4541	0.4601	W_H
25	0.7751	0.7878	W_H
50	0.8675	0.8990	W_H
75	0.8737	0.9203	W_H
100	0.9041	0.9406	W_H

The Weighted loss function under Exponential prior:W_E
L_E :The Linear loss function under Exponential prior
W_H: The Weighted loss function under Hypothetical prior
L_H: The Linear loss function under Hypothetical prior