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Hopf Bifurcation and Stability of an Improved Fluid Flow Model with Time Delay in Internet Congestion Control

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Abstract— **this paper mainly investigated a modified fluid flow model with time delay by using the control and bifurcation theory and discussed the effect of the communication delay on the stability. It is find that there exists a critical value of delay for the stability by using the communication delay as the bifurcation parameter. When the dealy value passes through the critical value, the equilibrium loses its stability and a Hopf bifurcation emerges. Besides, the linear stability of the model and the local Hopf bifurcation are studied and we derived the conditions for the stability and the existence of Hopf bifurcation at the equilibrium of the system. At last, some numerical simulation results are confirmed that the feasibility of the theoretical analysis.**

Index Terms— **Fluid flow, Communication delay, Hopf bifurcation, Stability, Numerical simulation**

I. Introduction

Nowadays, with the rapid advancement of science and technology, the Internet congetion control becomes a serious problem in practice use. When the required resources exceed the network capacity, it will cause congestion, which may lead to the loss of information and even the destruction of the whole system.Thus, the methods of Internet congestion control are very important [1-3]. Many congestion control mechanisms are developed to avoid the system congetion and collapse [4-7]. TCP and AQM are central to these congestion control mechanisms [8-10]. At present, Many researchers have studied the fluid flow model and obtained many conclusions [10-13].

In 2000, Misra et al. first proposed the fluid flow model of the differential equation of TCP/AQM [11]. Here we give a simplified edition of the model, which the TCP timeout mechanism is ignored. Such a model is described by the following nonlinear differential equations [11]:

$$\begin{cases} \dot{W} = \dfrac{1}{R(t)} - \dfrac{W(t)W(t-R(t))}{2R(t-R(t))} p(t-R(t)), \\ \dot{q} = N(t)\dfrac{W(t)}{R(t)} - C. \end{cases} \tag{1}$$

where $W(t)$ indicates the average value of TCP windowsize (packets), $q(t)$ represents the average queue length (packets), $N(t)$ is the number of TCP sessions and C is the queue capcity (packets/s), $p(g)$ is the probability function of a packet mark and $R(t)$ is the round-trip time which consists of the queuing delay and propagation delay. Both the queue length $q(t)$ and windowsize $W(t)$ are positive and bounded variables. When the loss probability is made roughly proportional to average queue length, namely $p(t) = Kq(t)$ [8] .so Eq. (1) becomes

$$\begin{cases} \dot{W} = \dfrac{1}{R} - \dfrac{W(t)W(t-R(t))}{2R(t-R(t))} Kq(t-R), \\ \dot{q} = N\dfrac{W(t)}{R} - C. \end{cases} \tag{2}$$

In [14], it shows that the Eq. (2) can be approiximated by

$$\begin{cases} \dot{W} = \dfrac{1}{R} - \dfrac{W(t)W(t-R)}{2R} Kq(t-R), \\ \dot{q} = N\dfrac{W(t)}{R} - C. \end{cases} \tag{3}$$

However, generally speaking, the queue delay is much smaller than the propagation delay. So queue delay can be ignored in the differential equation about the change of the windowsize. But, in the queue differential equation, since the change of queue length is directly related to the queue delay and a trifling variance of the queue will directly affect the propability of the packet mark and even the whole congestion state. So the delay cannot be completely ignorced. Hence, we propose a modified fluid flow model as follows:

$$\begin{cases} \dot{W} = \dfrac{1}{R} - \dfrac{W(t)W(t)}{2R} Kq(t-R) \\ \dot{q} = N\dfrac{W(t-R)}{R} - C \end{cases} \tag{4}$$

II. Stability and Local Hopf Bifurcation Analysis

In this section, we only discuss the problems of the Hopf bifurcation and stability for the unique positive equilibrium point (W_0, q_0). Then it satisfies

$$\frac{1}{R} - \frac{W_0 W_0}{2R} Kq_0 = 0; \qquad N\frac{W_0}{R} - C = 0. \tag{5}$$

That is

$$W_0 = \frac{RC}{N}; \qquad q_0 = \frac{2N^2}{R^2C^2K}. \tag{6}$$

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Let $x_1(t) = W(t) - W_0$, $x_2(t) = q(t) - q_0$. The linearation of system (4) at (W_0, q_0) is

$$\begin{cases} \dot{x}_1(t) = a_1 x_1(t) + a_2 x_2(t-R), \\ \dot{x}_2(t) = b_1 x_1(t-R). \end{cases} \quad (7)$$

where $a_1 = -\frac{2N}{R^2 C}, a_2 = -\frac{KRC^2}{2N^2}, b_1 = \frac{N}{R}.$

The correspoding characteristic equation of system (4) is as follows.

$$\lambda^2 - a_1 \lambda - a_2 b_1 e^{-2\lambda R} = 0 \quad (8)$$

Lemma 2.1 For the system (4), assume that $\omega_0 R_0 < \frac{\pi}{4}$ is satisfied.Then Eq.(8) has a pair of purely imaginary roots $\pm i\omega_0$ when $R = R_0$, where

$$\omega_0 = \sqrt{\frac{-a_1^2 + \sqrt{a_1^4 + 4a_2^2 b_1^2}}{2}},$$

$$R_0 = \frac{1}{2\omega_0} \arctan(-\frac{a_1}{\omega_0}).$$

Proof. Let $\lambda = i\omega\ (\omega > 0)$ is a solution of the characteristic equation (8), then

$$-\omega^2 - ia_1\omega - a_2 b_1 (\cos 2\omega R - i \sin 2\omega R) = 0.$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega^2 - a_2 b_1 \cos 2\omega R = 0 \\ -a_1 \omega + a_2 b_1 \sin 2\omega R = 0. \end{cases} \quad (9)$$

From (9) we obtain

$$\omega = \sqrt{\frac{-a_1^2 + \sqrt{a_1^4 + 4a_2^2 b_1^2}}{2}}$$

$$R = \frac{1}{2\omega} \left[\arctan(-\frac{a_1}{\omega}) + k\pi \right], \quad k = 0, 1, 2, \cdots.$$

Obviously, set $k = 0$, then

$$\omega_0 = \sqrt{\frac{-a_1^2 + \sqrt{a_1^4 + 4a_2^2 b_1^2}}{2}}, R_0 = \frac{1}{2\omega_0} \arctan(-\frac{a_1}{\omega_0}) \quad (10)$$

As a result, when $R = R_0$, the characteristic equation (8) have a pair of purely imaginary root. This completes the proof.

Lemma 2.2 Let $\lambda(R) = \alpha(R) + i\omega(R)$ be the root of (8) with $\alpha(R_0) = 0$ and $\omega(R_0) = \omega_0$ then we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{dR_0})^{-1}\Big|_{R=R_0} > 0$ is satisfied.

Proof. By differentiating both sides of Eq. (8) with regard to R and applying the implicit function theorem, we have

$$\frac{d\lambda}{dR}\Big|_{R=R_0} = \frac{-2a_2 b_1 \lambda e^{-2\lambda R_0}}{2\lambda - a_1 + 2a_2 b_1 R_0 e^{-2\lambda R_0}}$$
$$= \frac{-2a_2 b_1 \omega_0 \sin 2\omega_0 R_0 - i2a_2 b_1 \omega_0 \cos 2\omega_0 R_0}{(-a_1 + 2a_2 b_1 R_0 \cos 2\omega_0 R_0) + i(2\omega_0 - 2a_2 b_1 R_0 \sin 2\omega_0 R_0)}$$

then

$$\mathrm{Re}(\frac{d\lambda}{dR})\Big|_{R=R_0} = \frac{2a_2 b_1 a_1 \omega_0 \sin 2\omega_0 R_0 - 4a_2 b_1 \omega_0^2 \cos 2\omega_0 R_0}{(-a_1 + 2a_2 b_1 R_0 \cos 2\omega_0 R_0)^2 + (2\omega_0 - 2a_2 b_1 R_0 \sin 2\omega_0 R_0)^2} \quad (11)$$

Since $a_1 < 0, a_2 < 0, b_1 > 0$ and $0 < \omega_0 R_0 < \frac{\pi}{4}$, thus $\mathrm{Re}(\frac{d\lambda}{d\tau_0})^{-1} > 0$. The proof is completed.

Lemma 2.3 For Eq. (8), when $R < R_0$, all of his roots have negative real parts. The equilibrium (W_0, q_0) is locally asymptotically stable, and system (4) produces a Hopf bifurcation at the equilibrium (W_0, q_0) when $R = R_0$.

By applying the Hopf bifurcation theorem for delayed differential equation and the three lemmas [15], we have the following results.

Theorem 2.1. For system (4), the following conclusions hold:

If $R < R_0$, the equilibrium point is loaccly asymptotically stable.

If $R = R_0$,model (4) exhibits a Hopf bifurcation.

If $R > R_0$, then the equilibrium point is unstable.

III. Numerical Simulation

In this section, we present numerical results to confirm the analytical predictions obtained in the previous section. For a consistent comparision, we choose the same parameters as follows [16]

$$N = 50; \quad K = 0.001; \quad C = 1000.$$

From (10), we polt R_0 relationship with R in Fig.1. we kow that if $R < 0.179008$ then $R < R_0$ which indicates that the model (4) is stable. Namely, Hopf bifurcation occurs when $R_C = 0.179008$.

If we choose $R = 0.17$, which is a little less than R_0, from the corresponding analysis in Section 2, we get

$$W_0 = 3.4;\ q_0 = 173.01; a_1 = -3.46021; a_2 = -0.034; b_1 = 294.118;$$
$$\omega_0 = 2.38085;\ R_0 = 0.20332.$$

Since $R < R_0$, the equilibrium point (ω_0, q_0) of the system (4) is asymptotically stable proved by numerical simulations in Figs. 2-5.

If we change the delay R passes through to the critical value $R_C = 0.179008$, a Hopf bifurcation occurs, namely, there are periodic solutions bifurcating out from the equilibrium point (ω_0, q_0). We can choose $R = 0.19$, then a hopf bifurcation occurs as shown in Figs. 6-9, which indicate that there exists a stable limit cycle and is obviously consistent with the theorem in Section 2. Besides, when $R = 0.179008$, we get $R_0 = 0.179008$ and the periodic solutions occur from the equilibrium point (ω_0, q_0) which we can see in the Figs. 10-13.

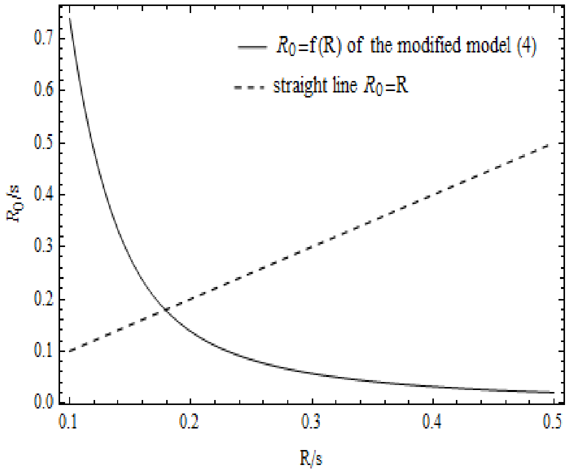

Figure 1. Relationship curve between R_0 and R.

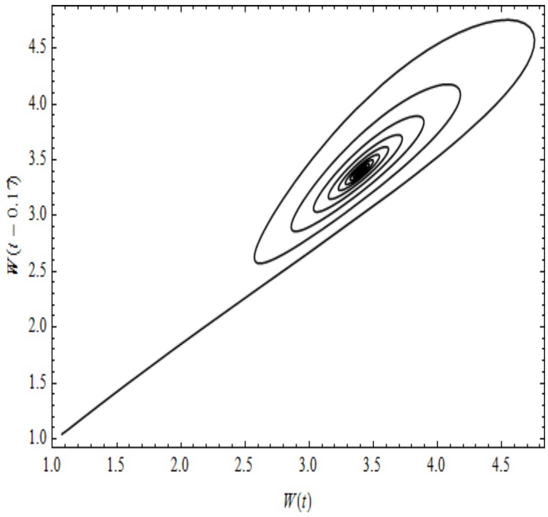

Figure 2. Phase plot of $W(t)$ with $R = 0.17$

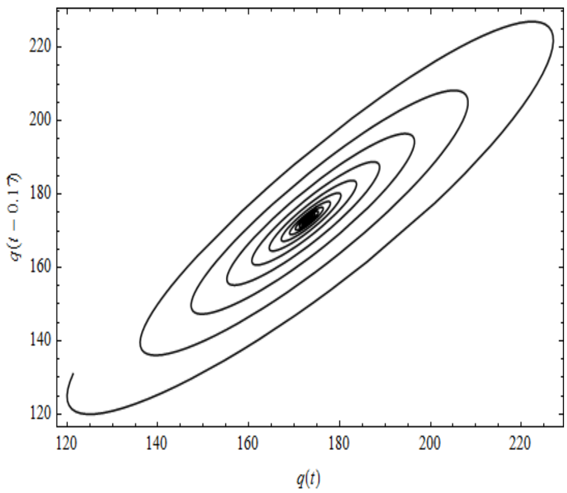

Figure 3. Phase plot of $q(t)$ with $R = 0.17$.

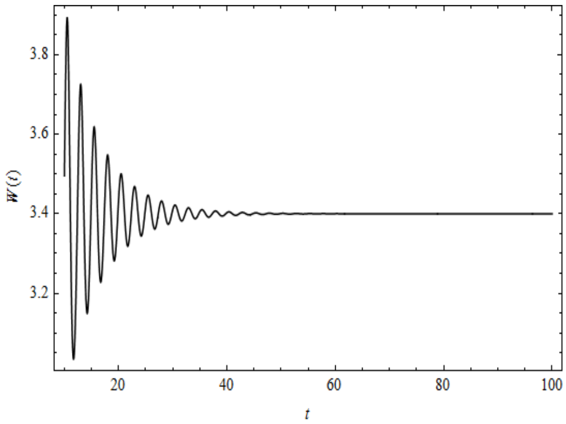

Figure 4. State plot of $W(t)$ with $R = 0.17$.

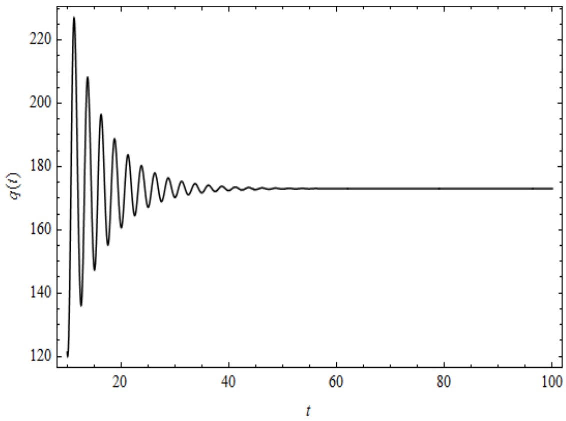

Figure 5. State plot of $q(t)$ with $R = 0.17$.

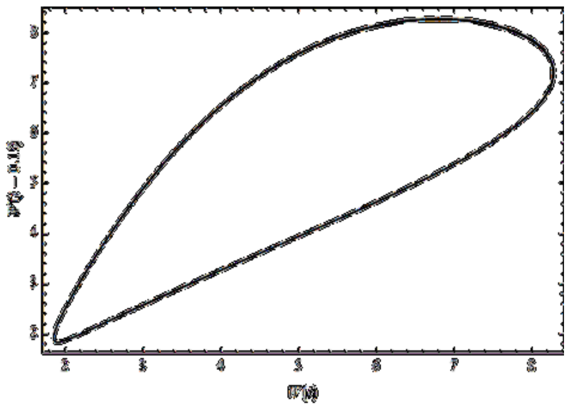

Figure 6. Phase plot of $W(t)$ with $R = 0.19$.

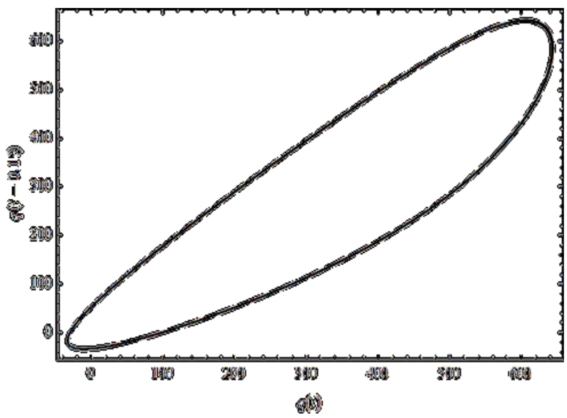

Figure 7. Phase plot of $q(t)$ with $R = 0.19$.

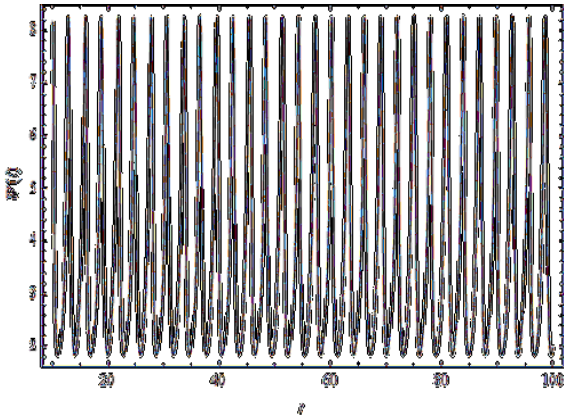

Figure 8. State plot of $W(t)$ with $R = 0.19$.

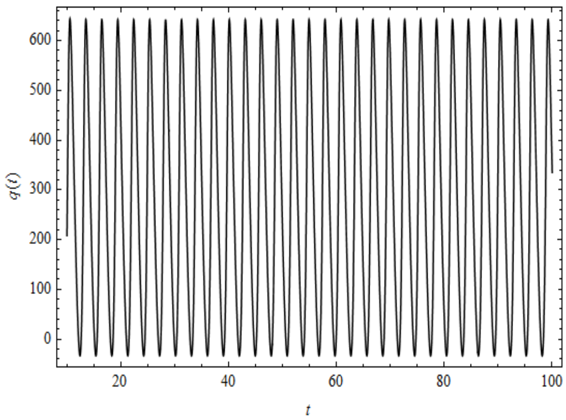

Figure 9. State plot of $q(t)$ with $R = 0.19$.

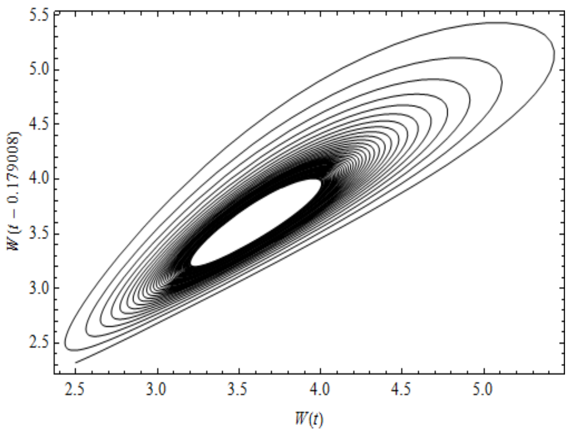

Figure 10. Phase plot of $W(t)$ with $R = 0.179008$.

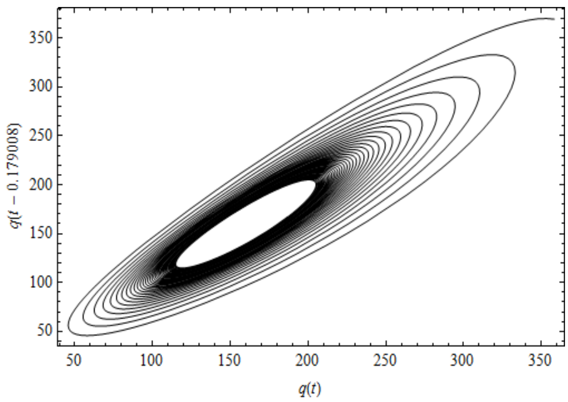

Figure 11. Phase plot of $q(t)$ with $R = 0.179008$.

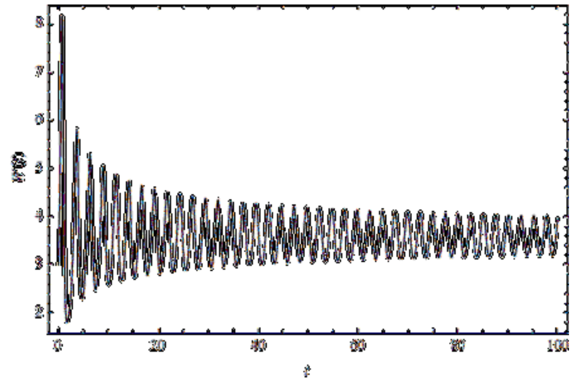

Figure 12. State plot of $W(t)$ with $R = 0.179008$.

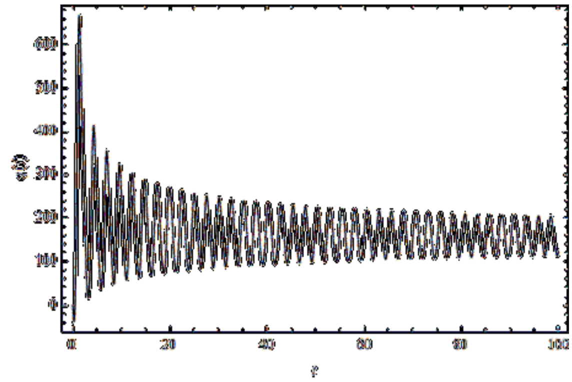

Figure 13. State plot of $q(t)$ with $R = 0.179008$.

IV. CONCLUSION

A modified fluid flow model of congestion control was studied by this paper. Through the above theoretical analysis, We have obtained the conditions that the system produces Hopf bifurcation. we also get that exists a critical value of communication delay for the stability of the system. When the delay of the system is less than this critical value the entire system is stable. we get that the system loses it stability and a Hopf bifurcation occurs when the communication delay passes through the critical value. The system will be congested or even collapse when the delay increases to large. Some computer simulation results have been presented to confirm the validity of the theoretical analysis.

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