Study on Stability and Convergence Speed of a Business Cycle System with Time Delay

Chunpu Zhang, Yanhui Zhai

***Abstract*— The stability and α-stability of the business cycle model with time delay are discussed. The linear stability and α-stability of the model are studied by using the geometric criterion of stability switching for time delay systems, and the differences and relations between the two stability are compared. Finally, numerical simulation is applied to verify the accuracy and effectiveness of the conclusion**

***Index Terms*— Time delay, Stability, α-Stability ,Equilibrium, Modified cycle model,Geometric criterion for stability switching of time delay systems**

I. THE ESTABLISHMENT OF THE MODEL

Ordinary differential equation is used to study the business cycle model, which is a research work gradually developed in recent years. In this paper, the stability criterion for time-delay systems is studied by using the geometric criterion of stability switching. In the literature[2], we get the model of business cycle model with bureau delays. As shown below

$$\begin{cases} x' = y \\ y' = -ax(t-\tau) - uy + qx^3 - vy^2 - vy^3 \end{cases} \quad (1.1)$$

Through the literature [2], we can get the equilibrium point of the system as shown below.

$E = (0,0)$.

And, the linearization of system (1.1) at E is

$$\begin{cases} x' = y \\ y' = -ax(t-\tau) - uy \end{cases} \quad (1.2)$$

II. STUDY ON MODEL STABILITY

According to the practical significance of the model, this paper only discusses the stability and stability convergence rate of the unique positive equilibrium E_0 .

From the literature [2], the characteristic equation of system (1.4) is as follows.

$$\lambda^2 + u\lambda + ae^{-\lambda\tau} = 0 \quad (2.1)$$

Considering $\tau = 0$ in first, the system (1.1) is the same as that in document [1], and its stability has been proved in [1]. When $\tau > 0$,We let $\lambda = i\omega(\omega > 0)$ is a solution of the characteristic equation (2.1), when and only when ω meet

$$-\omega^2 + iu\omega + a(\cos\omega\tau - i\sin\omega\tau) = 0 \quad (2.2)$$

The separation of the real and imaginary parts yields.

$$\begin{cases} -\omega^2 + a\cos\omega\tau = 0, \\ u\omega - a\sin\omega\tau = 0, \end{cases} \quad (2.3)$$

which lead to

$$Y(\omega,\tau) = \omega^4 + u^2\omega^2 - a^2 = 0 \quad (2.4)$$

The solution of equation (2.4), we can get

$$\omega^2 = \frac{-u^2 + \sqrt{u^4 + 4a^2}}{2} \quad (2.5)$$

$$\omega = \sqrt{\frac{-u^2 + \sqrt{u^4 + 4a^2}}{2}} \quad (2.6)$$

From literature[15],we can have get something .As shown below

$$\tan\theta = \frac{u}{\omega} \quad (2.7)$$

$$S_j = \tau - \tau_j = \tau - \frac{\theta + j2\pi}{\omega}, j = 0,1,2... \quad (2.8)$$

So $S_j = 0$, we can get the zero point τ_j . We can get the derivation of τ ,

$$S_j' = 1 \quad (2.9)$$

Obviously, we can get S_j is a monotone increasing function, so there is only one zero point instead of even zero points. We can see from the literature [15] that the zero point is the first stable switching point of equation group (2.2) and is the only one. That is,

$$S_0 = \tau - \frac{\theta}{\omega} = 0 \quad (2.10)$$

The only switching point of the equilibrium point of the dynamic system equation with delayed microorganisms is generated. We get the $\tau = \frac{\theta}{\omega}$, so we get a stable interval of $[0, \frac{\theta}{\omega})$.

III. NUMERICAL SIMULATION

Numerical simulation shows the system from stable to unstable complex transformation process. We give a concrete example to show the dynamic behavior of comprehensive national strength model. we take $a = -1.62, u = 2.7$ in the system (1.1). Through calculating of the Mathematica software, we obtain $\omega_0 \approx 0.586344, \tau_0 \approx 2.31431$. By stability switching theorem through time-delay systems , we can obtain that equilibrium E_0 is stable if $\tau = 1.65 < \tau_0$ (see Figure 1). By contrast, the equilibrium E_0 is unstable if $\tau = 2.65 > \tau_0$ (see Figure 3). Besides, when $\tau = \tau_0 = 2.31431$, the periodic solutions occur from the equilibrium E_0 (see Figure 2).

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However our analysis indicates that the dynamics of the comprehensive national strength model with time delay can be much more complicated than we may have expected. It is still interesting and inspiring to research.

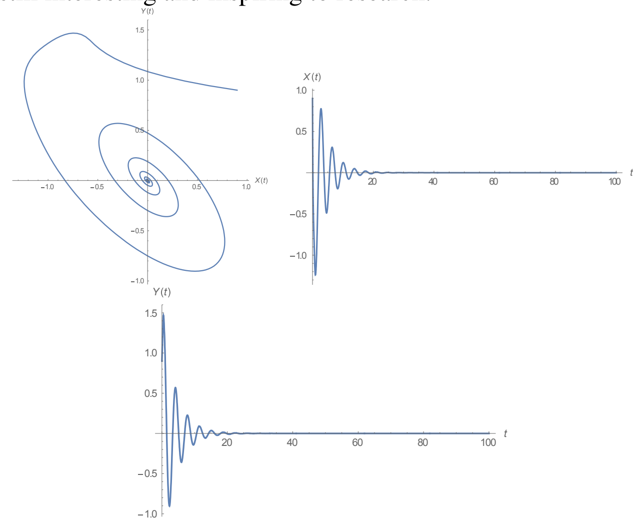


Figure 1 The equilibrium E_0 of system (1.4) is stable with $\tau=1.65<\tau_0$

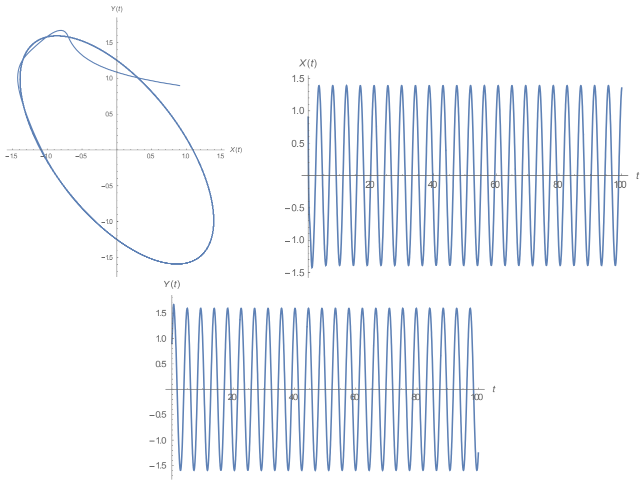


Figure 2 System (1.4) produce the periodic solutions with τ=2.31431=τ_0

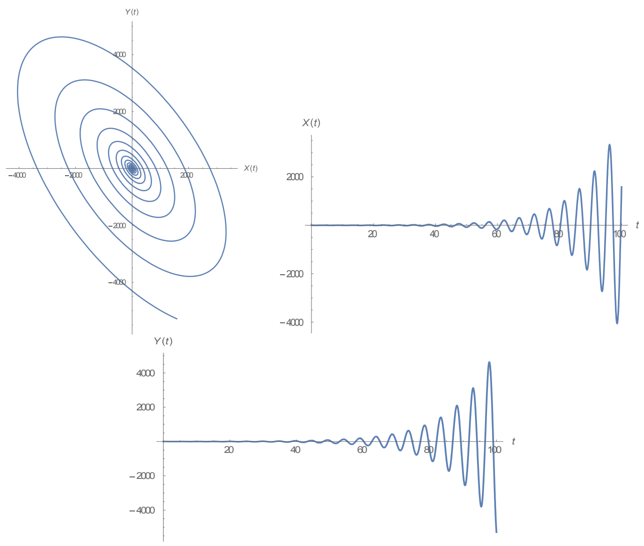


Figure 3 The equilibrium E_0 of system (1.4) is unstable with $\tau=2.65>\tau_0$

IV. STUDY ON THE STABILITY OF ALPHA OF THE MODEL

In literature [16], we get the definition of α- stability and the knowledge of its foundation for dynamical systems with time delay. The difference between stability and α- stability is also obtained. It is clear that the convergence rate is a problem, and the stability is slow to converge at the equilibrium point, and the convergence rate of α- stability is faster than the convergence rate of stability in the case of conditions.

In section (2), we obtain the characteristic equation (2.1) of the comprehensive national strength model , as follows,

$$\lambda^2+u\lambda+ae^{-\lambda\tau}=0$$

We study the α- stability of the model at equilibrium point through second sections and second cases in document [16].

Setting $\lambda=s-\alpha$,we can get:

$$Y(\lambda)=Y(s,\alpha)=s^2-(2\alpha-u)s+ae^{-s\tau}+\alpha^2-\alpha u=0 \quad (4.1)$$

Where, $p=2\alpha-u, q=\alpha^2-\alpha u$

We let $s=i\omega(\omega>0)$ is a solution of the characteristic equation (4.1), when and only when ω meet

$$-\omega^2-ip\omega+a(\cos 2\omega\tau-i\sin 2\omega\tau)+q=0 \quad (4.2)$$

The separation of the real and imaginary parts yields

$$\begin{cases} a\cos\omega\tau=\omega^2-q \\ a\sin\omega\tau=-p\omega \end{cases} \quad (4.3)$$

which lead to

$$Z(\omega,\tau)=\omega^4+(p^2-2q)\omega^2+q^2-a^2=0$$

(4.4)

The solution of equation (4.4), we can get

$$\omega^2=\frac{2q-p^2+\sqrt{(p^2-2q)^2-4(q^2-a^2)}}{2} \quad (4.5)$$

$$\omega=\sqrt{\frac{2q-p^2+\sqrt{(p^2-2q)^2+4(q^2-a^2)}}{2}}$$

(4.6)

From literature[15],we can have get something .As shown below

$$\tan\theta=\frac{p\omega}{q-\omega^2} \quad (4.7)$$

$$R_n=\tau-\tau_n=\tau-\frac{\theta+n2\pi}{\omega}, n=0,1,2... \quad (4.8)$$

Let's $R_n=0$ get zero points τ_j and we can get the derivative of τ.

$$R_n'=1 \quad (4.9)$$

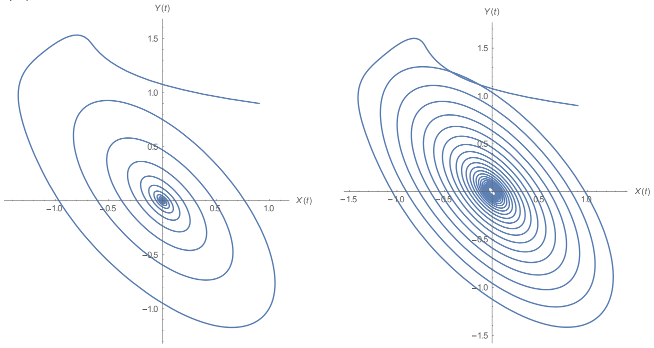
Obviously, we can get R_n and there is only one zero point instead of even zero points. We can see from the literature [15] that the zero point of R_n is the first stable switching point of the equation group, and it is unique. We remember it as τ_{01}.

V. NUMERICAL SIMULATION

According to the parameters in the numerical simulation of section third, let α=0.2 take a look at the change of system stability.

From the previous discussion, we can calculate by mathematical software, and solve the corresponding equation

(4.4), we can get (5.6) ω=0.527389 .It can also be calculated τ_{01}=2.037562 . In time, the model is α - stable. With the following figure 4, the system is stable at 1.84, such as Figure 4 (a), and the system is asymptotically stable when taking 2.05, such as Figure 4 (b), and the rate of convergence of Figure 4 (a) is faster than the rate of convergence of graph (b), then the model will gradually become unstable as above (3) as the value increases.



(a) τ=1.84 (b) τ=2.05

Figure 4 The time history diagram of the system

CONCLUSION

This paper apply delay in the business cycle model, which show rich dynamics behavior.The main work of this paper is as follows: first, we study the stability of the system (1.1) by the time-delay system stability geometric handover method. It is different from the Lyapunov functional method used in the literature [3-14]. The method of stability geometric switching used in this paper is very large in the discussion of local stability. Advantage. Second, the α -stability of the system (1.1) is studied by using the geometric switching method of time delay system stability, and then the convergence rate of stability and α- stability is compared. Finally, we verified the scientific nature of the theory through mathematical software. The results of this paper can be applied to our real life. We choose different parameters in different cases, and we can get the parameters directly to the conclusion of this paper. We can get τ and τ_{01}.So there are many undeveloped theories that need further exploration. We can also give different values to α and calculate different values of τ_{01} . Explore the convergence speed of the system, so that we can have an ideal control of the system according to our own needs in real life. This can achieve our ideal result.

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