Periodic Solution in a Predator-Prey System with Hassell-Varley Functional Response and Impulsive Harvesting

Fuyue Cheng, Dongwei Huang

***Abstract*— Predator and prey system in ecological biological are often interfered by human exploiting activities in impulsive patterns. For a predator-prey with Hapssell-Varley functional interaction model, firstly, the complex dynamic behaviors including stability of internal equilibrium and the existence, Hopf bifurcation, and the limit cycle are investigated roughly in this paper. Moreover, implementing multiple state feedback controls so that three order-1 periodic solutions driven are induced by disparate dynamics, respectively. Based on the method of successor function , we prove the existences of order-1 periodic solutions.**

***Index Terms*— Limit cycle; State feedback control; Order-1periodic solution; Successor function**

I. INTRODUCTION

Predators -prey models with Hassell-Varley functional response are one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [1]. A general predator-prey model with Hassell-Varley type functional response may discuss in the [2]. However, ecological biological are often interfered by human exploiting activities in impulsive patterns. Impulsive model has been used to multiple fields widely with threshold , such as virus control [3,4], ecological management and pest govern[5,6]. Chen et al.[7,8] has obtained Geometric theory of the impulsive system . Qualitative properties such as theuniqueness , existence and stability of order-1 periodic solutions are probed with the successor function [9].

The control with threshold has been used widely in the predator-prey model [10,11], in which function response with Hassell–Varley is absence. Meanwhile, the most of works have discussed a the periodic solution induced by function response with Holling II . On the other hand, the limit cycle has been widely applied in the continuous dynamic system. However, it's application in impulsive dynamical system was little due to the difficulty caused by discontinuities even though Tang has done some study [12, 13].

In this article, a predator-prey system model with Hapssell-Varley functional is formulated to discuss the two type periodic solutions induced by state feedback controls from disparate dynamics. The outline is instructed as follows: In Sect. 2, for the predator-prey model without pulse control, the existence and stability of the interior equilibrium are discussed. In Sect. 3, periodic solution provided by a stable interior equilibrium and the pulse strategy on limit cycle is discussed, respectively. In Sect. 4, numerical simulations are carried out to illustrate our results.

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II. A PREDATOR-PREY RESPONSE MODEL

Georgescu, P [2] presented a general model with Hassell-Varley type functional interaction. In this paper, it is assumed that $\gamma = 1/2$ describe the terrestrial predators that form a fixed number of tight groups.

Then a grazing system for predator (P) -prey (H) with Hassell-Varley interaction is given:

$$\begin{cases} \dfrac{dH}{dT} = aH\left(1-\dfrac{H}{K}\right) - \dfrac{cHP}{\left(mP^{1/3}+H\right)} \\ \dfrac{dP}{dT} = P\left(-D + \dfrac{fH}{\left(mP^{1/3}+H\right)}\right) \end{cases} \tag{2.1}$$

(1.) The parameters a, c, m, f, K and D are positive constants.

(2.) H and P denote, respectively, the densities of prey and predator at time T . The parameters K, a, D denote the constant carrying capacity, prey intrinsic growth rate, predator death rate, respectively.

(3.) The predation term $c/(mP^{1/3}+H)$ often represents a function response term. m is half saturation coefficient and c is the maximum consumption rate per herbivore.

Next, using $t \to aT, h \to H/K, p \to \alpha P$, we can rewrite the model (2.1) in the form

$$\begin{cases} \dfrac{dh}{dt} = h(1-h) - \dfrac{shp}{\left(h+p^{1/3}\right)} = F(h,p) \\ \dfrac{dp}{dt} = \delta p\left(-d + \dfrac{h}{\left(h+p^{1/3}\right)}\right) = G(h,p) \end{cases} \tag{2.2}$$

$$\alpha=\left(\frac{m}{K}\right)^2,\quad s=\frac{c}{a}\frac{1}{K}\left(\frac{K}{m}\right)^2,\quad \delta=\frac{f}{a},\quad d=\frac{D}{f}. \tag{2.3}$$

Before proceeding with the analysis of the model (2.1), we have to ensure that this system is well posed, i.e. its solutions are non-negative and bounded. However , from (2.2), it is easily observe that $\lim\limits_{(h,p)\to(0,0)} F(h,p)=\lim\limits_{(h,p)\to(0,0)} G(h,p)=0$. Define $F(0,0)=G(0,0)=0$, we can yield F and G are continuous on $R_+^2=\{(h,p)|h>0,p>0\}$.

2.1 Stability analysis

By simple calculation, we can obtain that system (2.2) has two equilibrium points $O(0,0)$ and $E(h^*,p^*)$, and O is a saddle point. For $E(h^*,p^*)$, its Jocobian matrix of E is

$$J(h_*,p_*)=\begin{bmatrix} h_*\left(-1+\dfrac{sp_*}{\left(h_*+p_*^{1/3}\right)^2}\right) & \dfrac{sh_*}{\left(h_*+p_*^{1/3}\right)^2}\left(h_*+\dfrac{2}{3}p_*^{1/3}\right) \\ \dfrac{\delta p_*^{3/4}}{\left(h_*+p_*^{1/3}\right)^2} & \dfrac{1}{3}\dfrac{-\delta h_* p_*^{1/3}}{\left(h_*+p_*^{1/3}\right)^2} \end{bmatrix}. \tag{2.4}$$

The characteristic equation at E is

$$\lambda^2+P_*\lambda+Q_*=0, \tag{2.5}$$

where

$$P_*=h_*\left(\frac{sp_*^{1/3}}{(h_*+p_*^{1/3})}-1-\frac{1}{2}\frac{\delta p_*^{1/3}}{(h_*+p_*^{1/3})^2}\right), \tag{2.6}$$

and

$$Q_*=\frac{1}{2}\delta\frac{h_*^2p_*^{1/3}}{(h_*+p_*^{1/3})^2}+\frac{1}{2}s\delta\frac{h_*p_*^2}{(h_*+p_*^{1/3})^4}+\frac{1}{2}s\delta\frac{h_*^2p_*^2}{(h_*+p_*^{1/3})^4}>0. \tag{2.7}$$

Thus, we can summarize this in the following proposition.

Proposition 2.1 For system (2.2), the stability of E is yielded.

(a) The unique interior equilibrium $E(h^*,p^*)$ is stable if $P_*<0$.

(b) E is unstable and system (2.2) has a limit cycle if $P_*>0$. Furthermore, Hopf bifurcation occurs providing $P_*=0$.

3. Periodicity driven by stable manifolds and limit cycle

3.1 Periodicity driven by stable manifolds

Let us assume that system (2.2) has stable interior equilibrium $E(h^*,p^*)$. This implies that all States drive to $E(h^*,p^*)$. It may be undesirable from biological or economical view. So in this section, we will take state feedback control strategy to reduce the quantities of prey and predator by αh and βp, respectively, when the prey reaches the level $p_1<p^*$. Thus, we consider the following model:

$$\begin{cases} \left.\begin{aligned} \frac{dh}{dt} &= h(1-h) - \frac{shp}{(h+p^{1/3})}, \\ \frac{dp}{dt} &= \delta p\left(-d + \frac{h}{(h+p^{1/3})}\right), \end{aligned}\right\} & p < p_1, \\ \left.\begin{aligned} \Delta h &= h(t^+) - h(t) = -\alpha h, \\ \Delta p &= p(t^+) - p(t) = -\beta p. \end{aligned}\right\} & p = p_1. \end{cases} \tag{3.1}$$

Where, $h(t^+)$ and $p(t^+)$ denote the level of predator and prey after a impulsive control is employed at time t.

Theorem 3.1 Suppose $P_* < 0$ and $p_1 < p^*$, $0 < \beta < 1.$, Then , for any $0 < \alpha < 1$,system (3.1) has an order-1 periodic solution.

Proof .For system (4.1), Impulse set and phase set can be denoted by M:$p = p_1$ and N: $p = (1-\beta)p_1$, respectively. Phase set N intersects with the $p-$nullcline $dp/dt = 0$ at point E, and there exists a trajectory passing through E which is tangent to phase set N: $p = (1-\beta)p_1$ (see Fig. 1(a)). This trajectory intersects the impulse set M as point $M_E(h_{M_A}, p_1)$. Then the impulsive function maps M_E to point N_E in N. Using the features of trajectory, point M_E locates on the upper right side of E, i.e., $h_{M_E} > h_E$. So, there is an α^* $(0 < \alpha^* < 1)$ satisfying $(1-\alpha^*)h_{M_E} = h_E$. This means, the phase point N_E and E are uniform. Then system (3.1) has an order-1 periodic solution.

If $0 < \alpha < \alpha^*$, then $(1-\alpha)h_{M_E} > (1-\alpha^*)h_{M_E}$, phase point N_E sets on the right side of E (see Fig. 1(b)). The successor function $F(E) = h_{N_E} - h_E > 0$. Denote the intersection point of $h-$nullcline $dh/dt = 0$ and the phase set N as point G. There exists a trajectory passing point G and which intersects impulsive set M at the point M_G. Then the impulsive function maps M_G to point N_G, and $F(G) = h_{N_G} - h_G < 0$. By the continuous character of successor function, system (3.1) has a point Q between E and G in phase set such that $F(G) = 0$. It means system (3.1) has an order-1 periodic solution.

If $0 < \alpha^* < \alpha < 1$, then $(1-\alpha)h_{M_E} < (1-\alpha^*)h_{M_E}$, and phase point N_E sets on the left side of E (see Fig. 1(c)). The successor function $F(E) = h_{N_E} - h_E < 0$. There exists a trajectory passing point N_E and which intersects impulsive set M at the point M_{G_1}. Then the impulsive function maps M_{G_1} to point N_{G_1}, and $F(G_1) = h_{N_{G_1}} - h_{N_E} < 0$. Using continuous character of successor function, there exists a point Q_1 between E and N_E in phase set such that $F(G_1) = 0$. It means system (4.1) has an order-1 periodic solution. The proof is completed.

(a) (b) (c)

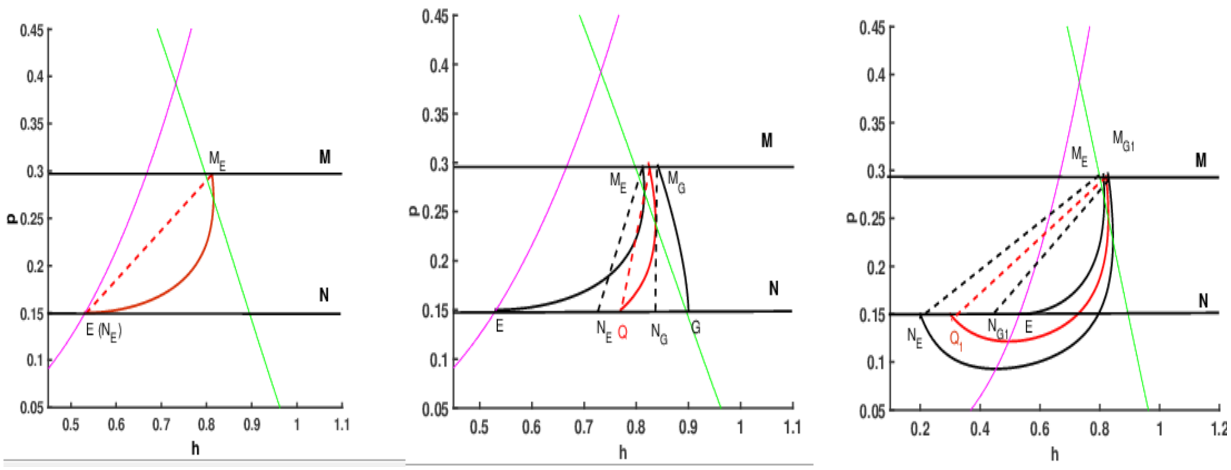

Fig1. Order-1 periodic solutions driven by stable manifolds. (a) Case of $\alpha=\alpha^*$.
(b) Case of $0<\alpha<\alpha^*$. (c) Case of $\alpha^*<\alpha<1$. Take $s=1, d=1/2,\ \delta=3,$
$p_1=0.3,\ \beta=0.5,\quad \alpha^*=0.34.$

Theorem 3.2 For (3.1), the order-1 periodic solution of system (3.1) is unique.
Choose arbitrary two points U, V in the phase set N and $0<h_U<h_v$ (see Fig. 2). There exist trajectories L_U and L_V passing U and V, and they intersect the impulse set M at M_U and M_V, respectively. M_U is located on the right of M_V. The impulsive function maps M_U to N_U and M_V to N_V, there $h_{N_U}=(1-\alpha)h_{M_U}$ and $h_{N_V}=(1-\alpha)h_{M_V}$. Then, $F(V)-F(U)=(h_{N_V}-h_V)-(h_{N_U}-h_U)<0$, which denotes the successor function is monotonically decreasing in the phase set. Thus, there exists only a point H satisfying $F(H)=0$. Hence, system (3.1) has unique an order-1 periodic solution. The proof is completed.

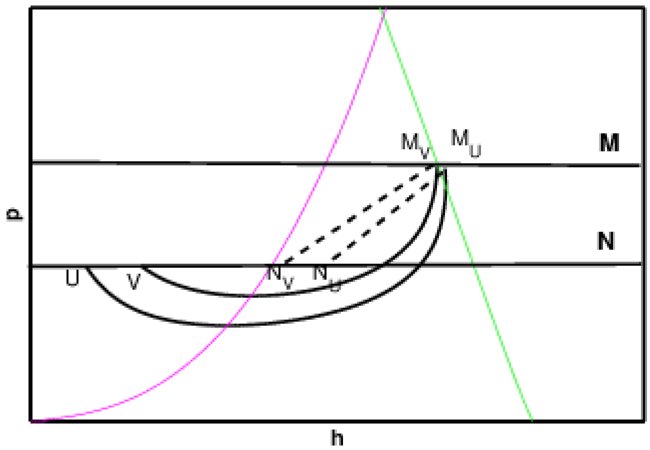

Fig2. Uniqueness of order-1 periodic solutions .

Lemma 3.1[4] If the integral along order-1 periodic solution of system(2.2) satisfies $\int_0^T \frac{\partial F}{\partial h}+\frac{\partial G}{\partial p}\,dt<0,$ the integral along order-1 periodic solution of system is stable.

Theorem 3.3 For (3.1), system (3.1) has a stable order-1 periodic solution if and only if $h>\frac{1}{2}$.

Proof. According to the equation (2.2), we have

$$\frac{\partial F}{\partial h}+\frac{\partial G}{\partial p}=1-2h-\delta d+\delta h-sp+\frac{2ph\left(s-\frac{1}{3}\delta p^{-2/3}\right)}{h+p^{1/3}}$$

Because the positive and negative of the above expression is uncertain, for based on the Dulac theorem, we can choose $B=\frac{1}{p}$. Then

$$\frac{\partial BF}{\partial h}+\frac{\partial BG}{\partial p}=\frac{1-2h}{p}-\frac{sp^{1/3}+\frac{1}{3}\delta hp^{-2/3}}{\left(h+p^{1/3}\right)\left(h+p^{1/3}\right)}<\frac{1-2h}{p}<0.$$

According to lemma 3.1, system (3.1) has a stable order-1 periodic solution. The proof is completed.

3.2 Periodicity driven by limit cycle

Let us assume that system (2.2) has unstable interior equilibrium $E(h^*,p^*)$. This implies that it emerges a limit cycle. In this subsection, through a state pulse control an order-1 periodic solution driven by this limit cycle is generated.

$$\begin{cases}\left.\begin{aligned}\frac{dh}{dt}&=h(1-h)-\frac{shp}{\left(h+p^{1/3}\right)},\\ \frac{dp}{dt}&=\delta p\left(-d+\frac{h}{\left(h+p^{1/3}\right)}\right),\end{aligned}\right\} & h<h_1,\\ \left.\begin{aligned}\Delta h&=h(t^+)-h(t)=\alpha h,\\ \Delta p&=p(t^+)-p(t)=-\beta p.\end{aligned}\right\} & h=h_1\end{cases}\qquad (3.2)$$

Theorem 3.4 Suppose $P_*>0$ and $h_1>h^*$, $0<\alpha<1$. , Then , for any $0<\beta<1$,system (3.1) has an order-1 periodic solution.

Proof . For system (3.2), denote the impulsive set M : $h=h_1$ and the phase set N : $(1+\alpha)h_1$. Phase set N intersects limit cycle at point A (see Fig. 3(a)). Denote the intersection point of trajectory of limit cycle and set M as M_A. Then the impulsive function maps V to N_A. There exists a β^* $(0<\beta^*<1)$ satisfying $\left(1-\beta^*\right)p_{M_A}=p_A$. This makes N_A and A uniform, i.e., $F(A)=0$. Hence, system (3.2) has an order-1 periodic solution.

If $\beta>\beta^*$, then $(1-\beta)p_{M_A}<\left(1-\beta^*\right)p_{M_A}$, phase point N_A locates on the below of the A, and successor function $F(A)=p_{N_A}-p_A<0$ (see Fig. 3(b)). There exists a trajectory of intersecting phase set N at the point B and which closes to the h axis. Denote the intersection point of trajectory and set M as M_B. Then the impulsive function maps M_B to point N_B, where point N_B locates on the above side of B. So, the successor function $F(B)=p_{N_B}-p_B>0$. Through the continuous character of the successor function, there exist point C between A and B in phase set satisfying $F(C)=0$. It means system 3.2 has an order-1 periodic solution .

If $\beta<\beta^*$, then $(1-\beta)p_{M_A}>\left(1-\beta^*\right)p_{M_A}$, phase point N_A locates on the above side of A, and successor function $F(A)=p_{N_A}-p_A>0$ (see Fig. 3(c)). There exists a trajectory of intersecting phase set N at the point B_1 and which should away from the h axis. Denote the intersection point of trajectory and set M as M_{B1}. Then the impulsive function maps M_{B1} to point N_{B1}, where point N_{B1} locates on the below side of B_1. So, the successor function $F(B_1)=p_{N_{B_1}}-p_{B_1}<0$. Through the continuous character of the successor function, there exist a point C_1 between A and B_1 in phase set satisfying $F(C_1)=0$. It means system (3.2) has an order-1 periodic solution . The proof is completed.

The unique and stable for system (3.2) can be found in above subsection; we omit the details.

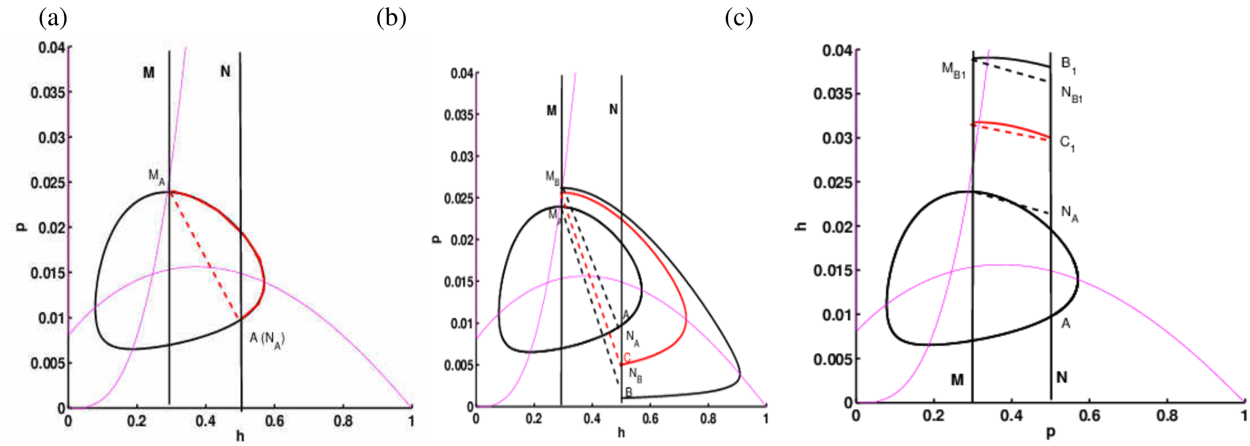

Fig3. Order-1 periodic solutions driven by stable manifolds. (a) Case of $\beta = \beta^*$. (b) Case of $\beta^* < \beta < 1$. (c) Case of $0 < \beta < \beta^*$. Take $s = 25$, $d = 1/2$, $\delta = 1$. $h_1 = 0.28$, $\alpha = 0.75$, $\beta^* = 0.59$.

DISCUSS

In this paper, for a predator-prey model with with Hapssell-Varley functional interaction , firstly, we investigated the existence and stability of the equilibrium, limit cycle, and Hopf bifurcation. Furthermore, multiple impulsive controls are discussed such that order-1 periodic solution provided by two dynamic behaviors. Based on the theorem of successor function, we prove the existences of order-1 periodic solution. Impulsive control is an effective scheme, but the control cost also is non-negligible, we will specially calculate cost in the future work.

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