Heteroclinic Bifurcation of a Predator-Prey System with Hassell-Varley Functional Response and Allee Effect

Fuyue Cheng, Dongwei Huang

***Abstract*— In this paper, we consider a Predator and prey system model with Hassell-Varley function and Alle effect. Firstly, we discuss the complex dynamic behaviors including the existence, stability of internal equilibrium. Moreover, implementing state feedback controls so that the Heteroclinic bifurcation is existed. Finally, choosing α as a control parameter, we obtain the sufficient conditions for system (3.1)the existence and uniqueness of order-1 periodic solution by using the geometry theory of semi-continuous dynamic systems.Based on the method of successor function , we prove the existences of order-1 periodic solutions.**

***Index Terms*— Heteroclinic bifurcation; Hassell-Varley functional responsel; Allee effect**

I. INTRODUCTION

Predators -prey models with Hassell-Varley functional response are one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance . A general predator-prey model with Hassell-Varley type functional response may discuss in the [1]. Menwhile, Predators -prey models with Alle effect is described by W[2, 3,4]. However, the analysis of predators -prey models with Hassell-Varley functional response and Alle effect is rare. Furthemore, ecological biological are often interfered by human exploiting activities in impulsive patterns. Impulsive model has been used to multiple fields widely with threshold , such as virus control [5,6], ecological management and pest govern[7,8]. Chen et al.[9.,10] has obtained Geometric theory of the impulsive system . Qualitative properties such as the uniqueness , existence and stability of order-1 periodic solutions are probed with the successor function [11].

The control with threshold has been used widely in the predator-prey model [12,13], in which function response with Hassell–Varley or Alle effect is absence. Meanwhile, the most of works have discussed a the periodic solution induced by function response with Holling II . On the other hand, the heteroclinic bifurcation has been widely applied in the continuous dynamic system. However, it's application in impulsive dynamical system was little due to the difficulty caused by discontinuities even though .

In this article, a predator-prey system model with Hapssell-Varley functional and Alle effect is formulated to discuss the heteroclinic bifurcation and the order-1 periodic solution induced by state feedback control. The outline is instructed as follows: In Sect. 2, for the predator-prey model without pulse control, the existence and stability of the interior equilibrium are discussed. In Sect. 3, heteroclinic cycle is provided by a stable and ustable mainfold, furthermore, when the heteroclinic cycle is broken, we discuss the order-1 periodic solution.

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Fuyue Cheng School of Science, Tianjin Polytechnic University, Tianjin, 300387, P.R. China

Dongwei Huang , Key experiment laboratory of textile, Tianjin Polytechnic University, Tianjin, 300387, P.R. China

II. A PREDATOR-PREY RESPONSE MODEL

P [2] presented a general model with Hassell-Varley type functional interaction. W[3] discribed the prey-predator model with Allee effect. In this paper, it is assumed that the predator-prey model with Allee effect and Hassell-Varley type functional interaction, which is descibed by $\gamma = 1/3$. Then a grazing system for predator (P) -prey (H) with Hassell-Varley interaction and Allee effect is given:

$$\begin{cases} \dfrac{dH}{dT} = aH\left(1-\dfrac{H}{K}\right)\left(H-\dfrac{B}{K}\right) - \dfrac{cHP}{\left(mP^{1/3}+H\right)} \\ \dfrac{dP}{dT} = P\left(-D+\dfrac{fH}{\left(mP^{1/3}+H\right)}\right) \end{cases} \tag{2.1}$$

(1.) The parameters a, c, m, b, f, K and D are positive constants.

(2.) H and P denote, respectively, the densities of prey and predator at time T. The parameters K, a, D denote the constant carrying capacity, prey intrinsic growth rate, predator death rate, respectively.

(3.) The predation term $c/(mP^{1/3}+H)$ often represents a function response term. m is half saturation coefficient and c is the maximum consumption rate per herbivore.

Next, using $t \to aT, h \to H/K, p \to \alpha P$, $b = B/K$, the model (2.1) can be rewritten by

$$\begin{cases} \dfrac{dh}{dt} = h(1-h)(h-b) - \dfrac{shp}{\left(h+p^{1/3}\right)} = F(h,p) \\ \dfrac{dp}{dt} = \delta p\left(-d+\dfrac{h}{\left(h+p^{1/3}\right)}\right) = G(h,p) \end{cases} \tag{2.2}$$

$$\alpha = \left(\frac{m}{K}\right)^2, \quad s = \frac{c}{a}\frac{1}{K}\left(\frac{K}{m}\right)^2, \quad \delta = \frac{f}{a}, \quad d = \frac{D}{f}. \tag{2.3}$$

2.1 Qualitative Analysis

From the view of biology, the dynamics of (2.2) are centered on the first closed quadrant R_2^+.

Theorem 2.1 We assume a positive and invariant box $\psi = \{(h, p \in R_+^2): 0 \le h \le Q, 0 \le p \le Q\}$ in R_2^+ such that all solutions of (2.2) with non-negative initial values approach ψ as $t \to +\infty$, where Q is a positive constant.

Proof. Obviously, $h(t) \ge 0$ and $p(t) \ge 0$ hold for all initial values in R_2^+. Again from the result

$\frac{dL}{dt}|L == \frac{dh}{dt}|h1 = -\frac{sp}{1+p^{1/3}} < 0$ on a straight line L : h− 1 = 0, we know that the trajectories pass through L from the right to the left. Define

$$V(h,p) = \delta h + p - \upsilon$$

with a undetermined $\vartheta > 0$. When $\vartheta > k$, the curve $V(h,p) = 0$ intersects with axis p and the line h =1 at the points E(ϑ,0) and F(k,$\vartheta - 1$) respectively. Then we have

$$\frac{dV}{dt}|EF = \delta h(1-h)(h-b) - \delta^2 pd - \delta pd.$$

Select a large enough $\vartheta(> k)$ such that $\frac{dV}{dt}|EF < 0$. Then the trajectories of system (2.2) cross the line EF from the upper right into the lower left.Define Q = max{1,ϑ−1} with $\vartheta > 1$. Then one gets a positive and invariant box ψ as a result that all solution with non-negative initial values draw near ψ when t → +∞. The proof is completed.

2.2 Stability analysis

In this subsection, we will research the qualitative characteristic of system (2.2) in R_2^+. The equilibria satisfy the following equation .

$$\begin{cases} \frac{dh}{dt} = h(1-h)(h-b) - \frac{shp}{(h+p^{1/3})} = 0 \\ \frac{dp}{dt} = \delta p\left(-d + \frac{h}{(h+p^{1/3})}\right) = 0 \end{cases} \tag{2.4}$$

By simple calculation, we can obtain that system (2.2) has two positive boundary equilibrium points $E_1(b,0)$ and $E_2(1,0)$, and a positive equilibrium $E_3(h^*, p^*)$. Next we will describe the stability of equilibria $E_1(b,0)$, $E_2(1,0)$ and $E_3(h^*, p^*)$. its Jocobian matrix is

$$J(h_*, p_*) = \begin{bmatrix} h_*(1-h_*) + (1-2h_*)(h_*-b) - \frac{shp_*^{3/4}}{(h_* + p_*^{1/3})^2} & \frac{sh_*}{(h_*+p_*^{1/3})^2}\left(h_* + \frac{2}{3}p_*^{1/3}\right) \\ \frac{\delta p_*^{3/4}}{(h_*+p_*^{1/3})^2} & \frac{1}{3}\frac{-\delta h_* p_*^{1/3}}{(h_*+p_*^{1/3})^2} \end{bmatrix}. \tag{2.5}$$

The characteristic equation at E is

$$\lambda^2 + P_*\lambda + Q_* = 0, \tag{2.6}$$

where

$$P_* = \mathrm{h}_*(1-\mathrm{h}_*) + (1-2h_*)(h_* - b) - \frac{\mathrm{sh}p_*^{3/4}}{\left(h_* + p_*^{1/3}\right)^2} - \frac{1}{3}\frac{\delta \mathrm{h}_* p_*^{1/3}}{(h_* + p_*^{1/3})^2}, \quad (2.7)$$

And

$$Q_* = \left(\mathrm{h}_*(1-\mathrm{h}_*) + (1-2h_*)(h_* - b) - \frac{\mathrm{sh}p_*^{3/4}}{\left(h_* + p_*^{1/3}\right)^2} \right) \frac{-\delta h_* p_*^{1/3}}{3\left(h_* + p_*^{1/3}\right)^2} - \frac{\delta p_*^{3/4} s h_* \left(h_* + \frac{2}{3} p_*^{1/3} \right)}{\left(h_* + p_*^{1/3}\right)^4}. \quad (2.8)$$

Through calculations, we get $Q_*(E_1) < 0, \quad Q_*(E_2) < 0$

Thus, $E_1(b, 0)$ and $E_2(1, 0)$ are saddle.

For system (2.2), the stability of E_3 is existed if and only if $P_* < 0$ and $Q_* > 0$ (see **Fig 1**).

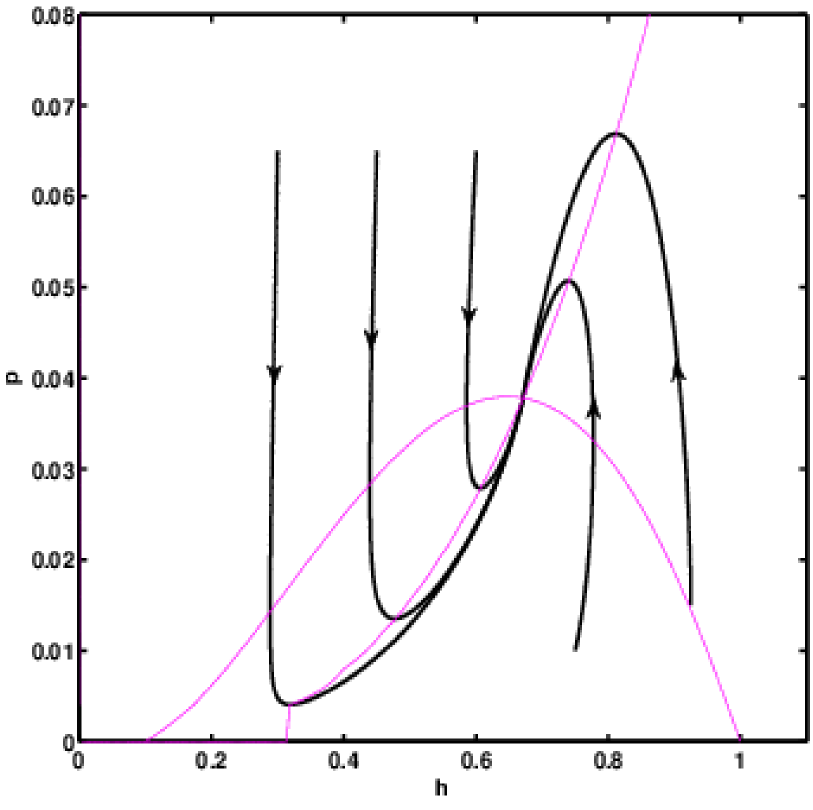

Fig 1. The stability of interior equilibrium d=2/3, r=1/3, δ =25,s=5,b=1/10.

3. PERIODICITY DRIVEN BY HETEROCLINIC CYCLE AND ORDER-1 PERIODIC SOLUTION

According to the above analysis,we know that the system (2.2) has two saddles E_1, E_2 and a stable positive point $E_3(h^*, p^*)$. It may be undesirable from biological or economical view. So in this section, we will take state feedback control strategy to reduce the quantities of prey and predator by αh and βp , respectively, when the prey reaches the level $p_1 < p^*$. Thus, we consider the following model:

$$\begin{cases} \left.\begin{aligned} \frac{dh}{dt} &= h(1-h)(h-b) - \frac{shp}{(h+p^{1/3})}, \\ \frac{dp}{dt} &= \delta p\left(-d + \frac{h}{(h+p^{1/3})}\right), \end{aligned}\right\} & p < p_1, \\ \left.\begin{aligned} \Delta h &= h(t^+) - h(t) = -\alpha h, \\ \Delta p &= p(t^+) - p(t) = -\beta p. \end{aligned}\right\} & p = p_1. \end{cases} \tag{3.1}$$

Where, $h(t^+)$ and $p(t^+)$ denote the level of predator and prey after a impulsive control is employed at time t.

Theorem 3.1 Suppose $P_* < 0$ and $p_1 < p^*$, $0 < \beta < 1.$, Then , for any $0 < \alpha < 1$,system (3.1) has an heteroclinic cycle if $\alpha = \alpha^*$.

Proof . For system (4.1), Impulse set and phase set can be denoted by $M{:}\,p = p_1$ and $N{:}\,p = (1-\beta)p_1$, respectively. For convenience, we denote the unstable manifold L_2 of E_2 intersects impulsive set $p = p_1$ at point A, and the stable manifold L_1 of E_1 intersects phase set N at point B. By the trajectory property of system (3.1), we have that L_1 and L_2 are above the vertical isocline $\frac{dh}{dt} = 0.$

The impulse function $f(h) = (1-\alpha)h$ is monotonically increasing about h and monotonically decreasing about α, and so there must exist a fixed value $\alpha^* \in (0,1)$ such that point A is mapped on to the point B after impulsive effect; that is, $f(h,\alpha^*) = (1-\alpha^*)h_A = h_B$. That mans, there must exist a fixed value $\alpha^* \in (0,1)$ such that point A is mapped on to the point B after impulsive effect. So, the the closed curve $AB \cup AE_1 \cup E_1E_2 \cup E_2A$ constitutes a cycle which passes through the saddles E_1 and E_2 (see **Fig. 2**). Therefore, the system (3.1) has an order-1 heteroclinic cycle.

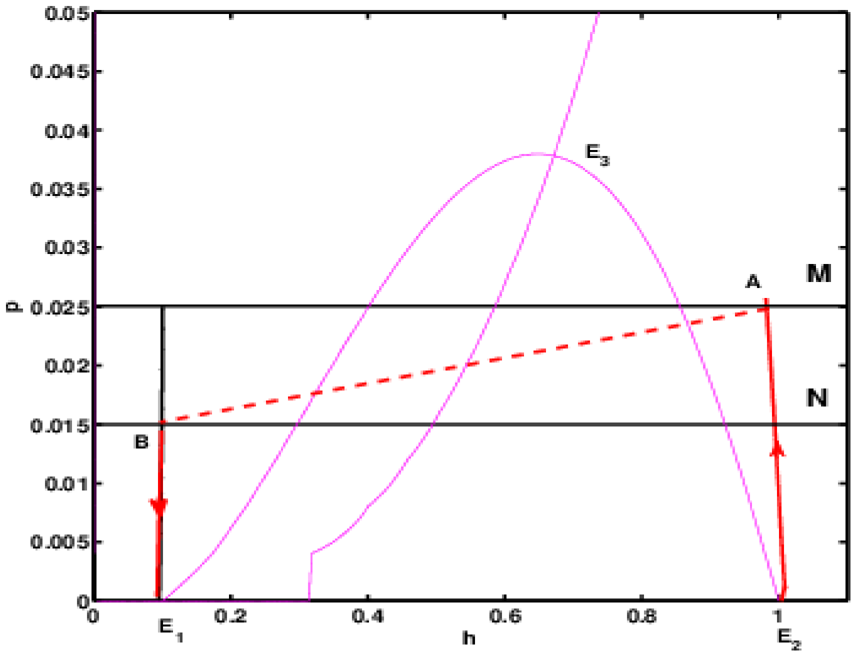

Fig. 2. The existence of order-1 heteroclinic cycle. d=2/3, r=1/3, δ =25,s=5,b=1/10.

If $0<\alpha<\alpha^*$, point A is mapped on to point A^+, and $(1-\alpha)h_{M_A}>(1-\alpha^*)h_{M_A}$. That means, the heteroclinic cycle is broken. Denote the intersection point of $h-$nullcline $dh/dt=0$ and the phase set N as the right point G and the left point E, respectively. First, there exists a trajectory passing point G and which intersects impulsive set M at the point M_G. Then the impulsive function maps M_G to point N_G, and $F(G)=h_{N_G}-h_G<0$. Second, there exists a trajectory passing point E and which intersects impulsive set M at the point M_E. Then the impulsive function maps M_E to point N_E, and $F(E)=h_{N_E}-h_E>0$. By the continuous character of successor function, system (3.1) has a point Q between E and G in phase set such that $F(Q)=0$. It means system (3.1) has an order-1 periodic solution.

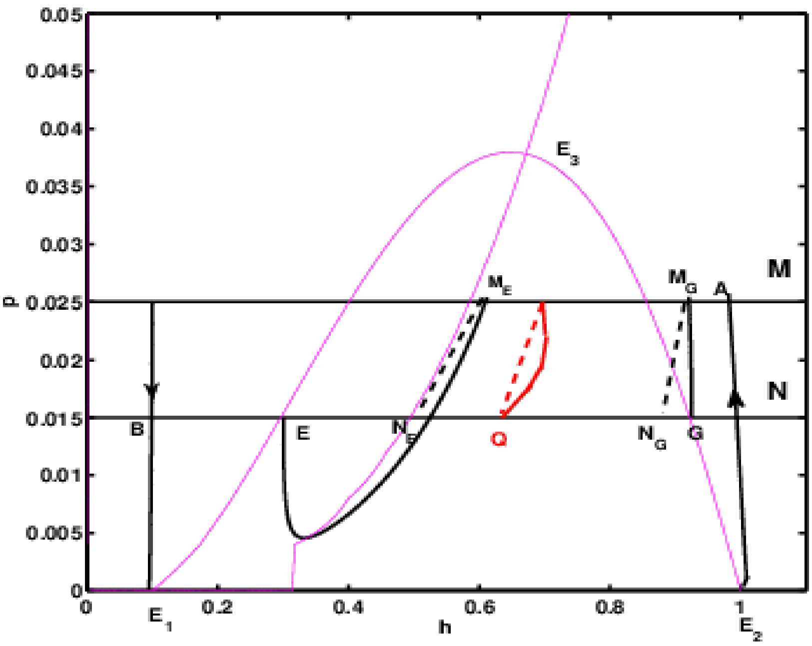

Fig. 3. The existence of order-1 periodic solution. d=2/3, r=1/3, δ =25,s=5,b=1/10.

DISCUSS

In this paper, for a predator-prey model with with Hapssell-Varley functional interaction and Alle effect . Firstly, we discuss the complex dynamic behaviors including the existence, stability of internal equilibrium. Moreover, implementing state feedback controls so that the Heteroclinic bifurcation is existed. Finally, choosing α as a control parameter, we obtain the sufficient conditions for system (3.1)the existence and uniqueness of order-1 periodic solution if the Heteroclinic cycle is broken . Impulsive control is an effective scheme, but the control cost also is non-negligible, we will specially calculate cost in the future work.

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