$H\infty$ Filter Analysis and Design for Systems with Time-varying Delay based on Novel Summation Inequalities

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***Abstract*— $H\infty$ filtering problem has been a particular important issue in networked control systems(NCSs) during recent years. As we know, the use of inequality technique when dealing with some cross terms is always a main source of conservatism. This paper firstly applies a Wirtinger-based inequality to $H\infty$ filtering problem for linear system with time delay. Considering the advance of novel summation inequality, we conclude a sufficient condition that guarantees system stable. Based on new delay- dependent bound real lemma(BRL) we obtained, a novel $H\infty$ filter design approach is proposed in terms of linear matrix inequality(LMI). Then, we address coupling between Lyapunov Functional and system matrix by congruence transformation. Finally, a $H\infty$ filter has been designed, which with much lower conservatism.**

***Index Terms*— NCSs; $H\infty$ filter; time-varying delay; delay partitioning; Lyapunov-Krasovskii functional; Wirtinger-based inequality**

I. INTRODUCTION

NCSs[1][2] lies in the intersection and combination of control theory, computer network and communication technology, which naturally brings resources sharing, mobile performing, easy installation and maintenance, high flexibility and reliability and so many other characteristics. NCSs has been a hot field both in theory study and practical engineering due to its outstanding virtues. On the other hand, network inevitably introduces some constraints into control systems, such as time delay and packet dropout which may drastically degrade the performance of system and even lead to instability. So it's certainly necessary to account time-delay during the process of system modeling and analysis.

$H\infty$ filter design[3-6] for system with time delay has been one of the hottest theme in control field over past decades.The objective of $H\infty$ filtering is to design a filter which satisfies the requirement that $H\infty$ norm of the filtering error system is minimal based on the assumption that the input noise signal is energy bounded. Regarding the existing results on $H\infty$ filter design for discrete-time delayed systems, much endeavor has been made to derive some delay- dependent criteria because delay-independent conditions[7-10] are much more conservative, especially for small time-delays. Anyway, how to address cross terms remains an important but meanwhile hard work. The author used famous Moon inequality[11] to bound the intersection terms in paper[12]. In 2000, Jesen inequality became the most popular method employed to integral or summation inequality scaling[13]. 2012, based on Wirtinger inequality[14] which includes Jesen inequality, papers [15] and [16] improved the results. 2015, Free-weighted matrices theory[17] was successfully displayed in the stability analysis of system with time-delay, which comprehensively considered system state, input, output and other terms. So it brings relatively lower conservatism. However, there is still a huge space to progress in the choice of L-K functional and in the process of dealing with cross terms.

Therefore, the new scaling technology is creatively adopt to filter design in this paper to solve $H\infty$ filtering problem. Accounting the fact that Wirtinger inequality is advanced than Jesen inequality or Free-weighted matrices method, eventually the conservatism is obviously reduced.

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II. PRELIMINARIES

A. *LEMMA 1[16]*

Given symmetric matrix $Z \in S_n^+$, and Any discrete time variable $x \in [0,d] \cap Z \to \mathbf{R}_n$, $d \geq 1$, the following inequality holds:

$$\sum_{i=-d+1}^{0} \delta^T(i) Z \delta(i) \geq \frac{1}{d} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}^T \begin{bmatrix} Z & 0 \\ 0 & 3r(d)Z \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \quad (1)$$

Where,

$$\delta(i) = x(i) - x(i-1),$$

$$r(d) = \frac{d+1}{d-1},$$

$$\theta_0 = x(0) - x(-d),$$

$$\theta_1 = x(0) + x(-d) - \frac{2}{d+1} \sum_{i=-d}^{0} x(i),$$

B. *LEMMA 2[16]*

m, n are both positive integers. Given symmetric matrices:

$$S_1 \in S_n^+, \ S_2 \in S_m^+,$$

If there exists $X \in \mathbf{R}^{n \times m}$, which satisfies

$$\begin{bmatrix} S_1 & X \\ X^T & S_2 \end{bmatrix} \geq 0, \quad (2)$$

Then for any scalar $\alpha \in (0,1)$, the following inequality holds:

$$\begin{bmatrix} \frac{1}{\alpha} S_1 & 0 \\ 0 & \frac{1}{1-\alpha} S_2 \end{bmatrix} \geq \begin{bmatrix} S_1 & X \\ X & S_2 \end{bmatrix}, \quad (3)$$

C. Lemma 3[18]

Assuming that there exist symmetric matrices P and G, the inequality:

$$-P + A^T PA < 0, \quad (4)$$

is equivalent to

$$\begin{bmatrix} -P & (GA)^T \\ GA & He(-G)+P \end{bmatrix} < 0, \quad (5)$$

D. PROBLEM FORMULATION

The linear discrete system with time-invariant delay can be described as follows:

$$\begin{aligned} &x(k+1) = Ax(k) + A_d x(k-d_k) + B\omega(k), \\ &y(k) = Cx(k) + C_d x(k-d_k) + D\omega(k), \\ &z(k) = Hx(k) + H_d x(k-d_k) + L\omega(k), \\ &x(k) = \varphi(k), k = -d_2, -d_2+1, \ldots, 0, \end{aligned} \quad (6)$$

where $x(k) \in \rfloor^n$ is the state vector, $y(k) \in \rfloor^q$ means the measured output, $z(k) \in \rfloor^p$ is the signal to be estimated, $\omega(k) \in \rfloor^l$ denotes the noise input which satisfies ω_- $\{\omega(k)\} \in L_2[0,\infty]$; And $x(k) = \phi(k)$, in which $k = -d_2, -d_2 +1, \ldots, 0$ is the given initial condition sequence; d_k means the time delay and is assumed to satisfy $1 \leq d_1 \leq d_k \leq d_2$ with d_1 and d_2 being the known lower and upper bounds of delay, respectively. $A, A_d, B, C, C_d, D, H, H_d, L$ are known time-constant system matrices.

The goal of this paper is to design a full-order linear asymptotically stable filter for system(4) with state-space realization of the form:

$$\begin{aligned} &x_F(k+1) = A_F x(k) + B_F y(k), \\ &z_F(k) = C_F x_F(k) + D_F y(k), \end{aligned} \quad (7)$$

where $x_F(k) \in \rfloor^n$ denotes the filter state; A_F, B_F, C_F, D_F are filter matrices to be determined.

Augmenting system (6) to include the filter states in system (7), we obtain the filtering error system as follows:

$$\begin{aligned} &\xi(k+1) = \overline{A}\xi(k) + \overline{A_d} K\xi(k-d_k) + \overline{B}\omega(k), \\ &e(k) = \overline{C}\xi(k) + \overline{C_d} K\xi(k-d_k) + \overline{D}\omega(k), \\ &\xi(k) = \left[\varphi^T(k) 0\right]^T, k = -d_2, -d_2+1, \ldots, 0, \end{aligned} \quad (8)$$

In which,

$$\overline{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix}, \overline{A_d} = \begin{bmatrix} A_d \\ B_F C_d \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix}, \overline{C} = \left[H - D_F C \quad -C_F\right],$$

$$\overline{C_d} = H_d - D_F C_d, \quad \overline{D} = L - D_F D,$$

$$\xi(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}, \quad e(k) = z(k) - z_F(k), \quad K = \left[I \quad 0\right],$$

Considering the fact that there is no control input in the original system model, the asymptotic stability of the filtering error system must be based on the assumption that system (6) is asymptotically stable.

The objective of this paper is to explore full-order $H\infty$ filters of form(7) meeting the following requirements:

1. the filtering error system in (8) is asymptotically stable;
2. under zero-initial conditions, for all nonzero $\omega \in l_2[0,\infty)$ and a given proper positive constant scalar γ ,the filtering error system in (8) guarantees

$$\|e\|_2 \leq \gamma \|\omega\|_2, \quad (9)$$

III. $H\infty$ PERFORMANCE ANALYSIS

First, we assume that filter's parameters are known, the following theorem presents sufficient conditions ensuring that estimation error system (8) is stable and has a prescribed $H\infty$ level γ in the FF domain of input noise. Based on the new conditions, the filter design method will be proposed later.

Theorem 1. For integers $1 \leq d_1 \leq d_2$, scalar $\gamma > 0$, provided that there exist real symmetric matrices $P > 0$, $Q_i = Q_i^T \geq 0$, $R = R^T \geq 0$, $S_j = S_j^T \geq 0$, and $X \in$ $^{2n\times 2n}$, where $i, j = 1, 2$, the close system (7) meets $H\infty$ application if the following inequality holds:

$$\begin{bmatrix} -P & 0 & 0 & P\Lambda_1 \\ * & -\Theta & 0 & \Theta\Lambda_2 \\ * & * & -I & \Lambda_3 \\ * & * & * & \Lambda_4 \end{bmatrix} < 0, \quad (10)$$

Where,

$$\Lambda_1 = \left[\overline{A} \quad 0 \quad \overline{A_d} \quad 0 \quad 0 \quad 0 \quad 0 \quad \overline{B}\right],$$

$$\Lambda_2 = \left[(A-I)K \quad 0 \quad A_d \quad 0 \quad 0 \quad 0 \quad 0 \quad B\right],$$

$$\Lambda_3 = \left[\overline{C} \quad 0 \quad \overline{C_d} \quad 0 \quad 0 \quad 0 \quad 0 \quad \overline{D}\right],$$

$$\Lambda_4 = \begin{bmatrix} \lambda & 0 & 0 & 0_{n\times 4n} & 0 \\ * & -Q_1 & 0 & 0_{n\times 4n} & 0 \\ * & * & -Q_2 - R & 0_{n\times 4n} & 0 \\ * & * & * & 0_{4n\times 4n} & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} + \Pi^T \begin{bmatrix} \mathbf{S}_1 & 0 & 0 \\ 0 & \mathbf{S}_2 & X \\ 0 & X & \mathbf{S}_2 \end{bmatrix} \Pi,$$

$$\lambda = -P + K^T(Q_1 + Q_2)K + (d_{12}+1)K^T RK,$$

$$\mathbf{S}_1 = diag\{S_1, 3r(d_1)S_1\}, \ \mathbf{S}_2 = diag\{S_2, 3S_2\},$$

$$r(d_1) = \frac{d_1+1}{d_1-1},$$

$$\Pi = \begin{bmatrix} M & & 0_{2n\times 3n} \\ 0_{2n\times n} & M & 0_{2n\times 2n} \\ 0_{2n\times 2n} & M & 0_{2n\times n} \end{bmatrix},$$

$$M=\begin{bmatrix} K & -I & 0 & 0 & 0 \\ K & I & 0 & 0 & -2I \end{bmatrix},$$

$$M=\begin{bmatrix} I & -I & 0 & 0 & 0 \\ I & I & 0 & 0 & -2I \end{bmatrix},$$

$$\Theta = d_1{}^2 S_1 + d_{12}{}^2 S_2,$$

$$\varsigma(k) \mid \left[\xi^T(k)\;\; x^T(k-d_1)\;\; x^T(k-d_k)\;\; x^T(k-d_2)\;\; \upsilon_1\;\; \upsilon_2\;\; \upsilon_3\right]^T,$$

$$\upsilon_1 \;\; \frac{1}{d_1+1}\sum_{i=k-d_1}^{k} x^T(i),\; \upsilon_2 \mid \frac{1}{d_k-d_1+1}\sum_{i=k-d_k}^{k-d_1} x^T(i),$$

$$\upsilon_3 \;\; \frac{1}{d_2-d_k+1}\sum_{i=k-d_2}^{k-d_k} x^T(i),$$

$$\varsigma_1(k) \rfloor [\varsigma^T(k) \quad \omega^T(k)]^T,\; \delta(k) \rfloor x(k)-x(k-1),$$

$$\delta(k+1) \mid x(k+1)-x(k)$$

$$=(A-I)x(k)+A_d x(k-d_k)+B\omega(k)=\Lambda_2\varsigma_1(k)$$

Proof: By virtue of new summation inequality technique, we choose the following L-K functional candidate:

$$V(k)=V_1(k)+V_2(k)+V_3(k)+V_4(k), \tag{11}$$

$$V_1(k)=\xi^T(k)P\xi(k), \tag{12}$$

$$V_2(k)=\sum_{i=k-d_1}^{k-1} x^T(i)Q_1x(i)+\sum_{i=k-d_2}^{k-1} x^T(i)Q_2x(i), \tag{13}$$

$$V_3(k)=\sum_{j=k-d_2+1}^{k-d_1+1}\sum_{i=k-1+j}^{k-1} x^T(i)Rx(i), \tag{14}$$

$$V_4(k)=d_1\sum_{j=-d_1+1}^{0}\sum_{i=k+j}^{k}\delta^T(i)S_1\delta(i)+d_{12}\sum_{j=-d_2+1}^{-d_1}\sum_{i=k+j}^{k}\delta^T(i)S_2\delta(i), \tag{15}$$

Taking the forward difference of $V(k)$,we have

$$\begin{aligned}\Delta V_1(k)&=\xi^T(k+1)P\xi(k+1)-\xi^T(k)P\xi(k)\\ &=\varsigma_1{}^T(k)\Lambda_1{}^T P\Lambda_1\varsigma_1(k)-\xi^T(k)P\xi(k)\end{aligned} \tag{16}$$

$$\begin{aligned}\Delta V_2(k)=&\sum_{i=k-d_1+1}^{k} x^T(i)Q_1x(i)+\sum_{i=k-d_2+1}^{k} x^T(i)Q_2x(i)\\ &-\sum_{i=k-d_1}^{k-1} x^T(i)Q_1x(i)-\sum_{i=k-d_2}^{k-1} x^T(i)Q_2x(i)\\ &=x^T(k)(Q_1+Q_2)x(k)-x^T(k-d_1)Q_1x(k-d_1)\\ &\quad -x^T(k-d_2)Q_2x(k-d_2)\\ &=\varsigma^T(k)\begin{bmatrix} K^T(Q_1+Q_2)K & 0 & 0 & 0_{n\times 4n}\\ * & -Q_1 & 0 & 0_{n\times 4n}\\ * & * & -Q_2 & 0_{n\times 4n}\\ * & * & * & 0_{4n\times 4n}\end{bmatrix}\varsigma(k)\end{aligned} \tag{17}$$

$$\begin{aligned}\Delta V_3(k)&=(d_2-d_1+1)x^T(k)Rx(k)-\sum_{i=k-d_2}^{k-d_1} x^T(i)Rx(i)\\ &\le (d_{12}+1)\xi^T(k)K^TRK\xi(k)-x^T(k-d_k)Rx(k-d_k)\\ &=\varsigma^T(k)\begin{bmatrix}(d_{12}+1)K^TRK & 0 & 0 & 0_{n\times 4n}\\ * & 0 & 0 & 0_{n\times 4n}\\ * & * & -R & 0_{n\times 4n}\\ * & * & * & 0_{4n\times 4n}\end{bmatrix}\varsigma(k)\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta V_4(k)=&\,\delta^T(k+1)(d_1{}^2S_1+d_{12}{}^2S_2)\delta(k+1)\\ &-d_1\sum_{i=k-d_1+1}^{k}\delta^T(i)S_1\delta(i)-d_{12}\sum_{i=k-d_2+1}^{k-d_1}\delta^T(i)S_2\delta(i)\\ =&\,\varsigma_1{}^T(k)\Lambda_2{}^T\Theta\Lambda_2\varsigma_1(k)-d_1\sum_{i=k-d_1+1}^{k}\delta^T(i)S_1\delta(i)\\ &-d_{12}\sum_{i=k-d_2+1}^{k-d_1}\delta^T(i)S_2\delta(i)\\ =&\,\varsigma_1{}^T(k)\Lambda_2{}^T\Theta\Lambda_2\varsigma_1(k)-d_1\sum_{i=k-d_1+1}^{k}\delta^T(i)S_1\delta(i)\\ &-d_{12}\sum_{i=k-d_2+1}^{k-d_1}\delta^T(i)S_2\delta(i)\end{aligned} \tag{19}$$

Applying lemma 1 to the last two items of $\Delta V_4(k)$, then we get:

$$\begin{aligned}-d_1\sum_{i=k-d_1+1}^{k}\delta^T(i)S_1\delta(i)\le -\varsigma^T(k)&\\ \begin{bmatrix} K & -I & 0 & 0 & 0 & 0 & 0\\ K & I & 0 & 0 & -2I & 0 & 0\end{bmatrix}^T\begin{bmatrix} S_1 & 0\\ 0 & 3r(d_1)S_1\end{bmatrix}&\\ \begin{bmatrix} K & -I & 0 & 0 & 0 & 0 & 0\\ K & I & 0 & 0 & -2I & 0 & 0\end{bmatrix}\varsigma(k)&\end{aligned} \tag{20}$$

$$\begin{aligned}-d_{12}\sum_{i=k-d_2+1}^{k-d_1}\delta^T(i)S_2\delta(i)\le -\frac{d_{12}}{d_k-d_1}\varsigma^T(k)&\\ \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0\\ 0 & I & I & 0 & 0 & -2I & 0\end{bmatrix}^T\begin{bmatrix} S_2 & 0\\ 0 & 3S_2\end{bmatrix}&\\ \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0\\ 0 & I & I & 0 & 0 & -2I & 0\end{bmatrix}\varsigma(k)&\end{aligned} \tag{21}$$

Define $\Psi=-d_1\sum_{i=k-d_1+1}^{k}\delta^T(i)S_1\delta(i)-d_{12}\sum_{i=k-d_2+1}^{k-d_1}\delta^T(i)S_2\delta(i)$, by adding (20) and (21), we have:

$$\Psi\le -\varsigma^T(k)\Pi^T\begin{bmatrix} S_1 & 0 & 0\\ 0 & \frac{d_{12}}{d_k-d_1}S_2 & 0\\ 0 & 0 & \frac{d_{12}}{d_2-d_k}S_2\end{bmatrix}\Pi\varsigma(k)$$

(22)

Where $\Pi=\begin{bmatrix} M & & 0_{2n\times 2n}\\ 0_{2n\times n} & M & 0_{2n\times n}\\ 0_{2n\times 2n} & & M\end{bmatrix}$,

From lemma 2, the following inequality holds:

$$\begin{bmatrix} S_1 & 0 & 0\\ 0 & \frac{d_{12}}{d_k-d_1}S_2 & 0\\ 0 & 0 & \frac{d_{12}}{d_2-d_k}S_2\end{bmatrix}\ge\begin{bmatrix} S_1 & 0 & 0\\ 0 & S_2 & X\\ 0 & X & S_2\end{bmatrix}, \tag{23}$$

According to Lyapunov stability theorem, $\Delta V(k)=\Delta V_1(k)+$

$\Delta V_2(k)+\Delta V_3(k)+\Delta V_4(k)<0$, arrange inequalities from (16) to (23), and assume $\omega(k)=0$, we have

$$\Delta V(k)\le \varsigma^T(k)\Phi\varsigma(k)<0, \qquad (24)$$

Where,

$$\Phi=\Lambda_2^T\Theta\Lambda_2+\Lambda_1^TP\Lambda_1+\Lambda_4,$$

$$\Lambda_4=\begin{bmatrix}\lambda & 0 & 0 & 0_{n\times 5n}\\ * & -Q_1 & 0 & 0_{n\times 5n}\\ * & * & -Q_2-R & 0_{n\times 5n}\\ * & * & * & 0_{5n\times 5n}\end{bmatrix}+\Pi^T\begin{bmatrix}S_1 & 0 & 0\\ 0 & S_2 & X\\ 0 & X & S_2\end{bmatrix}\Pi,$$

$$\Pi=\begin{bmatrix}M & 0_{2n\times 2n} & \\ 0_{2n\times n} & M & 0_{2n\times n}\\ 0_{2n\times 2n} & & M\end{bmatrix},$$

Then, accounting $H\infty$ performance index, we define

$$J\triangleq\sum_{k=0}^{\infty}\left[e^T(k)e(k)-\gamma^2\omega^T(k)\omega(k)\right], \qquad (25)$$

Under zero initial condition, due to $V(k)$ is Lyapunov functional, so $V(0)=0$ and $V(\infty)\ge 0$ are always valid. The following inequality (26) holds:

$$\begin{aligned}J&\le\sum_{k=0}^{\infty}\left[e^T(k)e(k)-\gamma^2\omega^T(k)\omega(k)\right]+V(\infty)-V(0)\\ &=\sum_{k=0}^{\infty}\left[e^T(k)e(k)-\gamma^2\omega^T(k)\omega(k)+\Delta V(k)\right] \qquad (26)\\ &\le\sum_{k=0}^{\infty}\varsigma_1^T(k)\Phi\varsigma_1^T(k)\end{aligned}$$

$$\Phi=\Lambda_3^T\Lambda_3+\Lambda_2^T\Theta\Lambda_2+\Lambda_1^TP\Lambda_1+\Lambda_4,$$

From (26), we find $\Phi<0$ is the sufficient condition of $J<0$, By Schur complement, $\Phi<0$ is equivalent to the inequality in (10). So we can conclude that filtering error system in (8) with $\omega(k)=0$ is asymptotically stable if the inequality in (10) holds. And for all $\omega(k)\in l_2[0,+\infty)$, meeting $H\infty$ performance index $\|e(k)\|<\gamma\|\omega(k)\|_2$.

$H\infty$ filter design

In this section, we will try to solve the $H\infty$ filter design problem based on the obtained BRLs.

Theorem 2. For system(8), if the LMI as (27) is solvable, $H\infty$ filter exists. If there exist matrices V_1, V_2, V_3, F, and symmetric matrices $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, $\bar{D}_F$, $P=\begin{bmatrix}P_1 & P_2\\ * & P_3\end{bmatrix}$, $Q=Q_i^T\ge 0$, $R=R^T\ge 0, S=S_j^T\ge 0$, $i,j=1,2$, $X=\begin{bmatrix}X_1 & X_2\\ * & X_3\end{bmatrix}$,

$$\begin{bmatrix}
\Omega_1 & \Omega_2 & 0 & 0 & \sigma_1 & \bar{A}_F & 0 & \sigma_2 & 0 & 0 & 0 & 0 & \sigma_3\\
* & \Omega_3 & 0 & 0 & \sigma_4 & \bar{A}_F & 0 & \sigma_5 & 0 & 0 & 0 & 0 & \sigma_6\\
* & * & \Omega_4 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & \alpha_3\\
* & * & * & -I & \beta_1 & -\bar{C}_F & 0 & \beta_2 & 0 & 0 & 0 & 0 & \beta_3\\
* & * & * & * & \bar{\Omega}_5 & -\bar{P}_2 & \Omega_6 & 0 & 0 & \Omega_{13} & 0 & 0 & 0\\
* & * & * & * & * & -\bar{P}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & \Omega_7 & \Omega_8 & \Omega_{10} & \Omega_{14} & \Omega_{16} & \Omega_{20} & 0\\
* & * & * & * & * & * & * & \Omega_9 & \Omega_{11} & 0 & \Omega_{17} & \Omega_{21} & 0\\
* & * & * & * & * & * & * & * & \Omega_{12} & 0 & \Omega_{18} & \Omega_{22} & 0\\
* & * & * & * & * & * & * & * & * & \Omega_{15} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & \Omega_{19} & \Omega_{23} & 0\\
* & * & * & * & * & * & * & * & * & * & * & \Omega_{24} & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -\gamma^2 I
\end{bmatrix}<0, \qquad (27)$$

Where,

$$\sigma_1=V_1^TA+\bar{B}_FC,\quad \sigma_2=V_1^TA_d+\bar{B}_FC_d,$$
$$\sigma_3=V_1^TB+\bar{B}_FD,\ \sigma_4=V_3^TA+\bar{B}_FC,$$
$$\sigma_5=V_3^TA_d+\bar{B}_FC_d,\quad \sigma_6=V_3^TB+\bar{B}_FD,$$
$$\alpha_1=F^T(A-I),\quad \alpha_2=F^TA_d,\quad \alpha_3=F^TB,$$
$$\beta_1=H-\bar{D}_FC,\quad \beta_2=H_d-\bar{D}_FC_d,\quad \beta_3=L-\bar{D}_FD,$$
$$\Omega_1=\bar{P}_1-V_1-V_1^T,\quad \Omega_2=\bar{P}_2-V_2-V_3,$$
$$\Omega_3=\bar{P}_3-V_2-V_2^T,\quad \Omega_4=\Theta-F-F^T,$$
$$\bar{\Omega}_5=-\bar{P}_1+Q_1+Q_2+(d_{12}+1)R+(3r(d_1)+1)S_1,$$
$$\Omega_6=(3r(d_1)-1)S_1,\ \Omega_7=(3r(d_1)+1)S_1+4S_2-Q_1,$$
$$\Omega_8=2S_2+X_1+2X_2+X_3,\quad \Omega_9=8S_2-2X_1+2X_3-Q_2-R,$$
$$\Omega_{10}=-X_1+X_3,\quad \Omega_{11}=2S_2+X_1-2X_2+X_3,$$
$$\Omega_{12}=4S_2,\quad \Omega_{13}=-6r(d_1)S_1,$$
$$\Omega_{14}=-6r(d_1)S_1,\quad \Omega_{15}=12r(d_1)S_1,$$
$$\Omega_{16}=-6S_2,\quad \Omega_{17}=-6S_2-2(X_2+X_3),$$
$$\Omega_{18}=2(X_2-X_3),\quad \Omega_{19}=12S_2,$$
$$\Omega_{20}=-2(X_2+X_3),\ \Omega_{21}=-6S_2-2(X_2-X_3),$$
$$\Omega_{22}=-6S_2,\ \Omega_{23}=4X_3,\ \Omega_{24}=12S_2,$$
$$A_F=V_2^{-1}\bar{A}_F,\quad B_F=V_2^{-1}\bar{B}_F,$$
$$C_F=\bar{C}_F,\quad D_F=\bar{D}_F.$$

Furthermore, the filter realization can be gained by:

$$A_F=V_2^{-1}\bar{A}_F,\quad B_F=V_2^{-1}\bar{B}_F,\quad C_F=\bar{C}_F,\quad D_F=\bar{D}_F. \qquad (28)$$

Proof. Given matrices G,F, by lemma 3, the gained BRL (10) is equivalent to the following inequality:

$$\Psi\triangleq\begin{bmatrix}P-G-G^T & 0 & 0 & G^T\Lambda_1\\ * & \Theta-F-F^T & 0 & F^T\Lambda_2\\ * & * & -I & \Lambda_3\\ * & * & * & \Lambda_4\end{bmatrix}<0, \qquad (29)$$

Noting that $\bar{P}>0$ and the LMI in (27) holds, so $\bar{P}_3>0$ and $-\bar{P}_3+V_2+V_2^T>0$ can be obtained respectively. It is obvious that $V_2+V_2^T>0$, i.e. V_2 is nonsingular. As a result, there always exist square and nonsingular matrices U and G_{22} satisfying $V_2=U^TG_{22}^{-1}U$. Construct matrix variables as follows:

$$J_1\triangleq\begin{bmatrix}I & 0\\ 0 & G_{22}^{-1}U\end{bmatrix},\ G_{12}=V_3U^{-1}G_{22}, \qquad (30)$$

$$G=\begin{bmatrix}V_1 & G_{12}\\ U & G_{22}\end{bmatrix},\ P=J_1^{-T}\begin{bmatrix}\bar{P}_1 & \bar{P}_2\\ * & \bar{P}_3\end{bmatrix}J_1^{-1}, \qquad (31)$$

$$A_F=U^{-T}\bar{A}_FU^{-1}G_{22},\ B_F=U^{-T}\bar{B}_F,$$
$$C_F=\bar{C}_FU^{-1}G_{22},\ D_F=\bar{D}_F,$$
$$J_2\triangleq diag\{J_1,I,I,J_1,I,I,I,I,I,I,I\},$$

we can see (25) and $\overline{P} > 0$ are equivalent to:

$$J_2^T \Pi J_2 < 0, \quad (32)$$

$$J_1^T P J_1 > 0, \quad (33)$$

So by applying congruence transformation to (32) and (33). It is readily seen that (27) is equivalent to (10). Therefore, from Theorem 2, it can be concluded that the filter with state-space realization (A_F, B_F, C_F, D_F) defined in (7) guarantees that the filtering error system in (8) is asymptotically stable with an H∞ noise attenuation level bound .By substituting $V_2 = U^T G_{22}^{-1} U$, we can get the filter state-space realization.

we rewrite (A_F, B_F, C_F, D_F) as

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} U^{-1}G_{22} & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} V_2^{-1}\overline{A}_F & V_2^{-1}\overline{B}_F \\ \overline{C}_F & \overline{D}_F \end{bmatrix} \begin{bmatrix} U^{-1}G_{22} & 0 \\ 0 & I \end{bmatrix}, \quad (33)$$

Considering U and G_{22} are non-singular, (A_F, B_F, C_F, D_F) is algebraically equivalent to $(V_2^{-1}\overline{A}_F, V_2^{-1}\overline{B}_F, \overline{C}_F, \overline{D}_F)$. Thus a state-space realization (A_F, B_F, C_F, D_F) of the prescribed filter can be obtained as (28). This completes the proof.

IV. SIMULATION EXAMPLES

In this section, we present several examples to illustrate the effectiveness and advantages of obtained results. All the calculations are finished by Matlab LMI toolbox.

A. *Example 1*

First，considering system(6) with specific parameters：

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.9 \end{bmatrix}, A_d = \begin{bmatrix} -0.1 & 0.15 \\ -0.1 & -0.15 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, D = 1,$$

$$H = \begin{bmatrix} 1 & 2 \end{bmatrix}, H_d = \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, L = -0.5,$$

And $d_1 = 6$, $d_2 = 8$, By theorem 2，we get optimal solution γ^*=9.9691 which is less conservative than existing results. When $d_2 = 9$, $H\infty$ filter realization

$$\left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array}\right]$$

are

$$\left[\begin{array}{cc|c} 0.1250 & 0.1086 & -0.2255 \\ 0.1380 & 0.3049 & -0.3008 \\ \hline 1.4573\times10^{-5} & -1.8674\times10^{-5} & 1.4699 \end{array}\right], \quad (34)$$

More detailed results are showed in table 1 :

d_2	7	8	9
[4]	7.5050	13.8580	∞
[19]theorem 4	6.0430	8.7038	13.9799
[19]theorem 5	6.2928	10.3752	29.9862
Theorem 2	5.9430	8.7332	15.3765

Tabel 1

B. *Example 2*

Considering system(8) with following state-space matrices：

$$A = \begin{bmatrix} 0.85 & 0.1 \\ -0.1 & 0.7 \end{bmatrix}, A_d = \begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, C = \begin{bmatrix} 0.2 & 2.5 \end{bmatrix},$$

$$C_d = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}, D = -1,$$

$$H = \begin{bmatrix} 0 & 2.2 \end{bmatrix}, H_d = \begin{bmatrix} 1.5 & -0.4 \end{bmatrix}, L = -0.1,$$

Let $d_1 = 2$, $d_2 = 6$, we get optimal solution γ^*=3.9910 which with lower conservativeness compared with existing papers. $H\infty$ filter parameters are：

$$\left[\begin{array}{cc|c} 0.1250 & 0.1086 & -0.2255 \\ 0.1380 & 0.3049 & -0.3008 \\ \hline 1.4573\times10^{-5} & -1.8674\times10^{-5} & 1.4699 \end{array}\right], \quad (37)$$

Similarly,more detailed simulation results can be found in table 2 which showed different results when time-delay upper and lower bound are different.

d_1	1	1	2	2
d_2	4	5	5	6
[5]	4.9431	6.1608	5.3551	6.7581
[19]	3.6545	4.6494	5.3458	6.7185
Theorem 2	2.7162	3.4553	4.7441	4.7742

Table 2

V. CONCLUSION

The delay-dependent results of $H\infty$ filtering problem for discrete-time systems with time-delay has been given in this paper.By constructing proper L-K functional and applying a novel Wirtinger-based inequality to address summation items produced by difference of LKF, two new delay-dependent BRLs have been present. The proposed $H\infty$ filter design procedures are formulated in terms of LMIs, which can be easily solved by the LMI toolbox in the MATLAB. Finally, two numerical examples have been presented. Due to Wirtinger inequality is advanced than Free-weighting matrix or Jensen inequality methods. The obtained results have much more advantage over the most existing results.

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REFERENCES

[1] Ke-You YOU, Li-Hua XIE, Survey of Recent Progress in Networked Control Systems, Acta Automatica Sinica, Volume 39, Issue 2, (2013) 101–117.

[2] Dan Zhang, Peng Shi, Qing-Guo Wang, Li Yu, Analysis and synthesis of networked control systems: A survey of recent advances and challenges, ISA Transactions, Volume 66, (2017) 376–392.

[3] Meng Zhang, Peng Shi, Zhitao Liu, Longhua Ma, Hongye Su, H∞ filtering for discrete-time switched fuzzy systems with randomly occurring time-varying delay and packet dropouts, Signal Processing, Volume 143, (2018) 320-327.

[4] Qiu J, Feng G, Yang J. Improved delay-dependent H∞ filtering design for discrete-time polytopic linear delay systems. IEEE Transactions on Circuits and Systems (II) 2008; 55(2):178-182.

[5] J. You , H. Gao, M V Basin. Further improved results on H∞ filtering for discrete time-delay systems[J]. Signal Processing, 2013, 93(7): 1845-1852.

[6] H. Gao, X. Meng, T. Chen, A new design of robust H-infinity filters for uncertain discrete-time state-delayed systems, in: 46th IEEE Conference on Decision and Control, 2007, pp. 5564–5569.

[7] C.E. de Souza, R.M. Palhares, P.L.D. Peres, Robust H∞ filter design for uncertain linear systems with multiple time-varying state delays, IEEE Trans. Signal Process. 49 (2001) 569–576.

[8] Z.Wang, F.Yang, D.W.C.Ho, X.Liu, Robust H∞ filtering for stochastic time-delay systems with missing measurements, IEEE Trans. Signal Process. 54 (2006) 2579–2587.

[9] R.M. Palhares, C.E. de Souza, P.L.D. Peres, Robust H∞ filtering for uncertain discrete-time state-delayed systems, IEEE Trans. Signal Process. 49 (2001) 1696–1703.

[10] S. Xu, Robust H∞ filtering for a class of discrete-time uncertain nonlinear systems with state delay, IEEE Trans. Circuits and Systems II 49 (2002) 1853–1859.

[11] Y.S. Moon, P. Park, W.H. Kwon, Y.S. Lee, Delay-dependent robust stabilization of uncertain state-delayed systems, Internat. J. Control 74 (2001) 1447–1455.

[12] H. Gao, C. Wang, A delay-dependent approach to robust H∞ filtering for uncertain discrete-time state-delayed systems, IEEE Trans. Signal Process. 52 (2004) 1631–1640.

[13] K. Gu An integral inequality in the stability problem of time-delay systems. Proceedings of the 39th IEEE Conference on Decision and Control ,Sydney, Australia, pp. 2805-2810, 2000.

[14] K. Liu, E. Fridman, Wirtinger's inequality and Lyapunov-based sampled-data stabilization, Automatica, vol 48 , pp. 102-108, 2012.

[15] A. Seuret and F. Gouaisbaut. Wirtinger-based integral inequality: Application to time-delay systems. Automatica,2013, 49(9):2860－2866.

[16] A. Seuret, F. Gouaisbaut and E. Fridman, "Stability of Discrete-Time Systems With Time-Varying Delays via a Novel Summation Inequality," in IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2740-2745, Oct. 2015.

[17] H. B. Zeng and Y. He and M. Wu and J. She Free-Matrix-Based Integral Inequality for Stability Analysis of Systems With Time-Varying Delay IEEE Transactions on Automatic Control vol.60, pp.2768-2772,2015

[18] Y. Zhao, J. Wu and P. Shi, "H∞ control of non-linear dynamic systems: a new fuzzy delay partitioning approach," in IET Control Theory & Applications, vol. 3, no. 7, pp. 917-928, July 2009.

[19] Li, X. W. Li, Z. C. Gao, H. J. “Further results on H-infinity filtering for discrete-time systems with state delay,” International Journal of Robust and Nonlinear Control, vol. 21, pp. 248 -270, 2011.