On The Application of Construction Method in Mathematical Analysis

Yang He, Yanhui Zhai

***Abstract*— In many ways to solve mathematical problems, "construction law" is a very clever innovative method, construction method is widely used in solving problems in mathematical analysis, can effectively solve some difficult problems in mathematics, in this paper, through mathematical analysis problems of concrete instance, the application of construction method in solving numerical problems is summarized. It is helpful to help us master the structural method, improve the quality of mathematics and cultivate the consciousness of innovation**

***Index Terms*— Construction method, Inequality, Sequence limit, Integral**

I. THE INTRODUCTION

In some math problems,sometimes a mathematical model is constructed to solve mathematical problems(Such as geometric figures, functions, equations, etc),to find some kind of intrinsic connection in the mathematical problem,through a certain connection in the problem, the mathematical problem can be made simple and clear, so as to play the role of transformation and bridge, and then find the way of thinking and method to solve the mathematical problem, this problem solving method is called construction method.[1] Construction method has been a common method of solving problems in high school mathematics, and now it is even more common in college mathematics.The construction method makes the mathematical problem solving break the convention, find a new way and get the solution skillfully.This paper mainly combines the concrete examples of mathematical analysis problems and the specific application of the categorization and summary construction method in solving the problem of number division.Let everybody master construction method proficiently, improve mathematical quality and cultivate innovation consciousness.

II. THE CONSTRUCTION OF THEOREM PROOF IN MATHEMATICAL ANALYSIS

The proof of many theorems in the textbook of mathematical analysis is solved by construction method, among which the proof of Lagrange mean value theorem is a typical example.

Case 1 Lagrange mean value theorem：If function $f(x)$ satisfies the following condition(ⅰ)$f(x)$ continuous on the closed interval $[a,b]$;（ⅱ）$f(x)$ is differentiable on the open interval (a,b), there is at least one point ξ in the open interval, such that $f'(\xi)=\frac{f(b)-f(a)}{b-a}$.

Analysis: Obviously, the above theorem becomes the Rolle mean value theorem when $f(a)=f(b)$, or you could say that the Rolle mean value theorem is a special case of the Lagrange mean value theorem.So we should construct a function $F(x)$ such that $F(a)=F(b)$, and we know from plane analytic geometry that the secant line through two points $A(a,f(a))$ and $B(b,f(b))$ is $y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$.

Poof: Do auxiliary function $F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$,so $F(a)=F(b)(=0)$

and F satisfy the other two conditions of the Rolle mean value theorem on $[a,b]$, so there is a point $\xi\in(a,b)$ makes $F'(\xi)=f'(\xi)-\frac{f(b)-f(a)}{b-a}=0$, that is $f'(\xi)=\frac{f(b)-f(a)}{b-a}$,So the Lagrange mean value theorem is proved.

Many theorems and propositions in mathematical analysis are difficult to prove directly, but if we change our thinking, construct new relations according to the known conditions, and convert the unknown into the known, the problem will be solved easily.

III. THE CONSTRUCTION OF AN ARGUMENT FOR THE EXISTENCE OF ROOTS

Case 2 If $r>0,n$ is positive integer, then positive number x_0 exists, so that $x_0^n=r$

(x_0 is called the n-th positive root of r （that's the arithmetic root）, called $x_0=\sqrt[n]{r}$)[2].

Analysis: To show that $x_0^n=r$, if you don't prevent the substitution, you have $x_0^n-r=0$,and it's not hard to see that x_0 is a root of $x_0^n=r$, x_0 is a zero of x^n-r.And then we immediately associate the zero point theorem, so we need the constructor $f(x)=x^n-r$,if it satisfies the "zero theorem" in its domain, it can be shown that there is at least root in this region[4].

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Yang He, School of Science, Tianjin Polytechnic University, Tianjin 300387, PR China
Yanhui Zhai, School of Science, Tianjin Polytechnic University, Tianjin 300387, PR China

Poof: If $f(x)=x^n-r$, because has $x^n\to+\infty when$ $x\to+\infty$,So there must be a positive number a makes $a^n>r$. Because $f(x)$ is continuous on $[0,\ a]$,and $f(0)=-r<0, f(a)=a^n-r>0$, so $f(0)f(a)<0$,So the zero point theorem tells us that there is at least one point $x_0\in(0,a)$ makes $f(x_0)=x_0^{\,n}-r=0$,that is $x_0^{\,n}=r$.

In order to prove the existence of roots, the paper analyzes and synthesizes the set conditions and the conclusion to be proved(It usually involves moving the terms of both sides of the equation to one side, and it usually maps to the roots of the equation),at this point, the corresponding equation can be constructed to combine the quantitative relation given by the set conditions into a new concrete relation, so as to communicate the internal relation between the condition and the conclusion of the mathematical problem and solve the problem.

IV. Construction of Inequality Proof

Case 3 So given $0<x<y<\frac{\pi}{2}$, let's find $\frac{\sin x}{\sin y}>\frac{x}{y}$.

Analysis: The difficulty in proving this inequality is that there is no direct connection between the two sides,and if we can go through the observation and we can go through the observation and find that the inequality can be transformed to $\frac{\sin x}{x}>\frac{\sin y}{y}$,Now, the inequality is symmetric on both sides, and it's easy to think about whether you can use the function $f(t)=\frac{\sin t}{t}$,as long as you can prove that $f(x)>f(y)$ is true in the case of $0<x<y<\frac{\pi}{2}$.

Proof: Assuming that $f(t)=\frac{\sin t}{t}, t\in\left(0,\frac{\pi}{2}\right)$, so $f'(t)=\frac{t\cos t-\sin t}{t^2}$, making $g(t)=t\cos t-\sin t$, $t\in\left(0,\frac{\pi}{2}\right)$, so $g'(t)=-t\sin t<0, t\in\left(0,\frac{\pi}{2}\right)$, so $g(t)$ is monotonically decreasing on $t\in\left(0,\frac{\pi}{2}\right)$, so $g(t)<g(0)=0$,thus $f'(t)=\frac{t\cos t-\sin t}{t^2}<0$, so $f(t)$ is monotonically decreasing on $t\in\left(0,\frac{\pi}{2}\right)$ and because $0<x<y<\frac{\pi}{2}$, so there is $f(x)=\frac{\sin x}{x}>f(y)=\frac{\sin y}{y}$, that is $\frac{\sin x}{\sin y}>\frac{x}{y}$.

The most traditional way to prove inequality problems is to do the difference or quotient,the above problems can also be solved by conventional methods, but in the calculation process, there will be some expressions that are difficult to simplify to judge the positive and negative,that is to say, some problems will be complicated by conventional methods, and we need to have a high degree of observation and analysis ability, and more importantly, divergent thinking.By using analogy to construct a new formula, the general case is to first transfer the term, get a similar formula, so as to construct the corresponding function,Second, take the derivative.Finally, solving the problem with monotone.

V. The Construction of Limit Problem of Sequence

Case4 Let $\{x_n\}$ be defined as follows: $x_1=a, x_2=b, x_n=\frac{1}{2}(x_{n-1}+x_{n-2})(n=3,4,5\cdots\cdots)$, to calculate $\lim\limits_{n\to\infty} x_n$.

Analysis: Given the specific values of the first two terms of the sequence, and A recursive relationship starting from the third term, the final result is the limit of x_n ,The key question is we need to know what is the exact expression of x_n ,so we need a finite set of conditions to construct it properly.

Solution: Construct series $\sum\limits_{k=1}^{\infty}(x_k-x_{k-1})$ ($x_0=0$) , the specific definition of $\{x_k-x_{k-1}\}$ is as follows :

$$x_2-x_1=b-a=(-\frac{1}{2})^0(b-a),$$

$$x_3-x_2=\frac{1}{2}(x_2+x_1)-x_2=(-\frac{1}{2})^1(b-a)$$

$$x_4-x_3=\frac{1}{2}(x_3+x_2)-x_3=(-\frac{1}{2})^2(b-a),\cdots\cdots,$$

$$x_k-x_{k-1}=\frac{1}{2}(x_{k-1}+x_{k-2})-x_{k-1}=(-\frac{1}{2})^{k-2}(b-a)\cdots\cdots,$$

so

$$\lim_{n\to\infty}x_n=\sum_{k=1}^{\infty}(x_k-x_{k-1})=a+(b-a)\sum_{k=2}^{\infty}(-\frac{1}{2})^{k-2}=\frac{1}{3}(a+2b).$$

Sequence knowledge is closely related to series knowledge.According to the quantitative relationship and structural characteristics of the conditions in the problem,constructing a series,then, according to the theory of series, the problem is transformed under the new relationship, so as to simplify the problem and finally solve the problem.

VI. Structure of Integral Problem

Case 5 To calculate $I=\int_0^{+\infty}\frac{\sin x}{x}e^{-ax}dx(a>0)$

Analysis: At first glance, the expression of the integrand of the definite integral is very complicated, and direct calculation will have great problems.But if you look closely, you'll see $\frac{\sin x}{x}e^{-ax}=\int_0^1 e^{-ax}\cos xtdt$, and then the integral is distorted.Thus, the complicated calculation process can be avoided.

Solution:Because $\frac{\sin x}{x}e^{-ax}=\int_0^1 e^{-ax}\cos xtdt,$ so

$$I=\int_0^{+\infty}\frac{\sin x}{x}e^{-ax}dx=\int_0^{+\infty}dx\int_0^1 e^{-ax}\cos xtdt,\ \int_0^{+\infty}e^{-ax}dx$$

converges when $a > 0,$ and when $0 \le t \le 1,$ we have $\left|e^{-ax}\cos xt\right| \le e^{-ax}.$ $\int_0^{+\infty} e^{-ax}\cos xtdt$ converges uniformly over $0 \le t \le 1$ .So

$$I = \int_0^{+\infty}\frac{\sin x}{x}e^{-ax}dx = \int_0^1 dt\int_0^{+\infty} e^{-ax}\cos xtdx = \int_0^1\left|\frac{e^{-ax}(-a\cos xt + t\sin tx)}{a^2+t^2}\right|_0^{+\infty} dt$$

$$= \int_0^1\frac{-a}{a^2+t^2}dt = -\arctan\frac{1}{a}.$$

In general, for the definite integral $\int_a^b f(x)dx$ when $f(x)$ is more complicated,finding a way to introduce a continuous function $\Phi(x,y),$ over the rectangular region $a \le x \le b$, $c \le y \le d$, such that $f(x) = \int_c^d \Phi(x,y)dy$,so let's construct $\int_a^b f(x)dx = \int_a^b dx\int_c^d \Phi(x,y)dy$,If $\int_a^b dy\int_c^d \Phi(x,y)dx$ is easier to calculate than $\int_a^b dx\int_c^d \Phi(x,y)dy$, then the definite integral $\int_a^b f(x)dx$ can be obtained by calculating $\int_a^b dy\int_c^d \Phi(x,y)dx$, thus simplifying the operation process.

CONCLUSION

Construction method is a very important idea to solve problems, which has a wide range of applications in mathematical analysis,due to the limitation of the length of the article and my limited ability, I cannot introduce them one by one.However, through the above induction and arrangement, it is not difficult to find that construction method can simplify and simplify problems in solving mathematical problems, so as to solve problems quickly and give people a sense of winning by surprise.One other thing we have to pay attention to is that constructivism is very useful in solving problems, there is no qualitative method, we need to be flexible and adopt the appropriate method according to the specific problem.

General steps of construction method: (ⅰ) Read the questions carefully and analyze in depth the conditions and conclusions given by the questions. (ⅱ) According to the given conditions and the characteristics of the conclusion, related mathematical knowledge, to reconstruct the relationship.

(ⅲ) Thus using the auxiliary elements, find the basic form of the required structure, clear thinking. (ⅳ) Solve the problem by using the correct construction method and making correct and detailed solutions.

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REFERENCES

1. D.L.Chen ,On the construction method in solving mathematical problems [J]. Mathematics teaching and research.2012,20;
2. Mathematics department, east China normal university. Mathematical analysis [M]. Higher education press.2014;
3. X.P.Chen, Discussion on solving problems with structural method [J]. Mathematics learning and research.2011,7;
4. h.t.Wu,P.He, Application of structure method in solving problems of higher mathematics. Journal of changchun university of science and technology (higher education edition) [J].2009,84 -- 85,99;
5. X.Y.Shi,Application research of structural method in higher mathematics. Journal of chifeng university (natural science edition) [J].2015, 11-13;
6. X.Huang , Selected theory of mathematical methodology [M]. Chongqing university press.1995