Study on the Stability Convergence Rate of Fluid Flow Congestion Control Model with Time Delay

Yang He, Yanhui Zhai

***Abstract*— This paper studies the fluid flow congestion control model with time delay by using the switching geometric criterion method of stability for systems with time delay, discusses the stability and α -stability of the model, and verifies the accuracy and validity of the conclusions by numerical simulation, finally compares the differences and connections between the two stability**

***Index Terms*— Delay; Stability; α - stability; Characteristic equation; Stability switching geometric criterion method**

I. INTRODUCTION

Mathematical methods are widely used in biology and population dynamics. There are also time-delay cases, and in some of the problems, there are phenomenon literatures [1-6] with time-delay related parameters.This makes the stability analysis more difficult. The *Lyapunov* function method is usually used, that is, the direction and stability of *Hopf* bifurcation can be given by constructing the central fashion and using the norm method.According to literature [6], *Beretta* and *Kuang* proposed a simple and feasible "geometric criterion method".The method is described below: Eigen equations for delay differential equations of the following form

$$P_n(\lambda,\tau)+Q_m(\lambda,\tau)\exp(-\lambda\tau)=0 \quad (1)$$

Here $P_n(\lambda,\tau)=\sum_{k=0}^{n}p_k(\tau)\lambda^k$; $Q_m(\lambda,\tau)=\sum_{k=0}^{m}q_k(\tau)\lambda^k$

Where $n,m\in N_0, n>m, p_k(\bullet), q_k(\bullet), P_n, Q_m$ is continuously differentiable with respect to τ .

Let $\lambda=i\omega,(\omega>0)$ substitute into equation (1) to get the discriminant function $S_n(\tau)=\tau-\tau_n(\tau),\tau\in I,n\in N_0$ of the stability region. By calculating the zero point τ value of S_n and according to its corresponding discriminant theorem, the stability interval of the equilibrium point corresponding to equation (1) can be obtained.

We generally want to converge to the equilibrium point more quickly, so that we can get the desired result easily by controlling the parameters,In reference [7], the fast convergence problem of a class of time-delay systems is discussed, that is, the α -stability problem of the time-delay dynamic system. The following is the introduction of the α -stability of the time-delay dynamic system:

Consider the characteristic equation of a class of time-delay systems in the following form

$$P(\lambda)+Q(\lambda)e^{-\lambda\tau}=0 \quad (2)$$

$P(\lambda)$ and $Q(\lambda)$ here are both polynomials of λ , and $\partial(P(\lambda))>\partial(Q(\lambda))$ .If all characteristic roots of equation (2) satisfy $\mathrm{Re}(\lambda)<0$, then the zero solution of the system is asymptotically stable.If $\lambda=s-\alpha$,and α are positive real Numbers, equation (2) can be changed to

$$P(s-\alpha)+Q(s-\alpha)e^{\alpha\tau}e^{-s\tau}=0 \quad (3)$$

Write equation (3) as

$$P(s,\alpha)+Q(s,\tau)e^{-s\tau}=0 \quad (4)$$

Here $P(s,\alpha)=P(s-\alpha)$ is independent of τ , and $Q(s,\tau)=Q(s-\alpha)e^{\alpha\tau}$ is independent of τ .If all the roots satisfy $\mathrm{Re}(s)<0$, then $\mathrm{Re}(\lambda)=\mathrm{Re}(s)-\alpha<-\alpha<0$, for A large positive number α , the solution of the system starting from the equilibrium attachment can quickly converge to the equilibrium point.Therefore, if the equilibrium point is α -stability, it will be asymptotically stability. The reverse is not necessarily true.

In this paper, based on the improved fluid flow congestion control model with time delay presented in reference [8], the convergence rate of the stability of the fluid flow congestion control model with time delay is studied by using the stability geometric switching method, that is, the stability and α -stability of the model, and the differences and relations between the two kinds of stability are compared.

II. MODEL ESTABLISHMENT

Internet congestion in practice is a very serious problem, when the required resources exceed the capacity of the network will produce congestion, it may lead to the loss of information or even the destruction of the whole system[9]-[11],therefore, it is of great significance to study the problem of Internet congestion.In order to study the dynamic characteristics of the Internet congestion control model and further understand the phenomenon of Internet congestion control, researchers have done a lot of related work and proposed a lot of related models.In literature [8], an improved fluid flow

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Yang He, School of Mathematical Science, TianGong University, Tianjin 300387, P.R China
Yanhui Zhai, School of Mathematical Science,TianGong University, Tianjin 300387, P.R China

congestion control model with time delay is established:

$$\begin{cases} \dot{W} = \dfrac{1}{\tau} - \dfrac{W(t)\dot{W}(t)}{2\tau} Kq(t-\tau) \\ \dot{q} = N\dfrac{W(t-\tau)}{\tau} - C \end{cases} \quad (5)$$

Where $W(t)$ represents the size of the congestion window of transmission control protocol (TCP) (unit: packet); $q(t)$ represents the router's average queue length (unit: packet); $N(t)$ represents the number of TCP sessions; C represents link capacity (unit: packet/second); $\tau(t)=\tau+q(t)/C$ represents round-trip communication delay, including transmission delay and queuing delay; $q(t)$ and $W(t)$ are both positive bounded variables.

III. Stability Analysis

Set the equilibrium point of model (5) as $E=(W_0,q_0)$, and then

$$\begin{cases} \dfrac{1}{\tau} - \dfrac{W_0 W_0}{2\tau} Kq_0 = 0 \\ N\dfrac{W_0}{\tau} - C = 0 \end{cases} \quad (6)$$

Can solve $W_0=\dfrac{\tau C}{N}, q_0=\dfrac{2N^2}{\tau^2 C^2 K}$ (7)

Let $u_1=W(t)-W_0, u_2=q(t)-q_0$, then the linear approximation system obtained after model (5) is linearized at the equilibrium point $E=(W_0,q_0)$ is

$$\begin{cases} \dot{u}_1(t)=a_1u_1(t)+a_2u_2(t-\tau) \\ \dot{u}_2(t)=b_1u_1(t-\tau) \end{cases} \quad (8)$$

in which $a_1=-\dfrac{2N}{\tau^2 C}, a_2=-\dfrac{KRC^2}{2N^2}, b_1=\dfrac{N}{\tau}$.

The characteristic equation of equation (8) is $\lambda^2 - a_1\lambda - a_2b_1e^{-2\lambda\tau}=0$ (9)

When $\tau=0$, the literature [8] has proved that $\operatorname{Re}(\lambda)<0$, that is, the model is stable at the equilibrium point.Next, observe whether the real part of some characteristic roots of eigen equation (9) will increase to zero or even become positive as the delay value τ increases.

When $\tau>0$, the literature [8] has also proved that the model is stable at the equilibrium point, but its proof process is relatively complex. Now it is proved by using the time-delay system stability switching geometric criterion method.

Take $\lambda=i\omega$, substitute it into equation (9), and you get

$$\omega^2+ia_1\omega+a_2b_1(\cos 2\omega\tau - i\sin 2\omega\tau)=0 \quad (10)$$

Separate the real part from the imaginary part and you get

$$\begin{cases} \omega^2+a_2b_1\cos 2\omega\tau=0 \\ a_1\omega-a_2b_1\sin 2\omega\tau=0 \end{cases} \quad (11)$$

Equation $F(\omega,\tau)=\omega^4+a_1^2\omega^2-a_2^2b_1^2=0$ (12) can be obtained from equation (11)

you get $\omega^2=\dfrac{-a_1^2+\sqrt{a_1^4+4a_2^2b_1^2}}{2}$ (13)

that is $\omega=\sqrt{\dfrac{-a_1^2+\sqrt{a_1^4+4a_2^2b_1^2}}{2}}$ (14)

From literature[12],you can get function

$$\sin\theta=\frac{a_1\omega}{a_2b_1}, \cos\theta=-\frac{\omega^2}{a_2b_1} \quad (15)$$

So $\tan\theta=-\dfrac{a_1}{\omega}$ (16)

From literature[12],you know

$$S_j=\tau-\tau_j=\tau-\frac{\theta+j2\pi}{2\omega}, j=0,1,2\cdots\cdots \quad (17)$$

Let's take $S_j=0$, , find the zero and call it τ_j, and the derivative of S_j with respect to τ is $\dfrac{d(S_j)}{d\tau}=1>0$, and of course S_j is A monotone increasing function, so S_j has and only has one zero, not even zero,according to literature [12], the zero point of S_0 is the first and unique stability switching point of model (5).

Therefore $S_0=\tau-\tau_0=\tau-\dfrac{\theta}{2\omega}=0$ that is $\tau_0=\dfrac{\theta}{2\omega}$, generates the unique stability switching point of the equilibrium point of the flow congestion control model with time delay, so we can get that the stability interval of this model is $\left[0,\dfrac{\theta}{2\omega}\right)$.

IV. Numerical Simulation of Model Stability

This section selects appropriate parameters and verifies the feasibility of the above theoretical analysis results through numerical simulation.The parameter selected in this article is : $N=50, K=0.001, C=1000$. Substitute them into model (5) and get the following expression:

$$\begin{cases} \dot{W} = \dfrac{1}{\tau} - \dfrac{W(t)\dot{W}(t)}{2\tau} 0.001q(t-\tau) \\ \dot{q} = 50\dfrac{W(t-\tau)}{\tau} - 1000 \end{cases} \quad (18)$$

It is easy to calculate $\tau_0=0.179008$ by using mathematical software, so model (18) is asymptotically stable when $\tau\in[0,\tau_0)$, When $\tau=\tau_0$, the system produces *Hopf* branch, that is the periodic solution. When $\tau\in(\tau_0,\infty)$, it is unstable.

Choose $\tau=0.17<\tau_0$ and you get $W_0=3.4, q_0=173.01, a_1=-3.46021, a_2=-0.034, b_1=294.118$,

$\omega_0 = 2.38085$,at this point, the system is asymptotically uniformly stable at the equilibrium point, as shown in figure 1;Similarly, when $\tau = 0.179008 = \tau_0$, the system produces *Hopf* branches, as shown in figure 2;When $\tau = 0.19 > \tau_0$, the system is unstable at the equilibrium point, as shown in

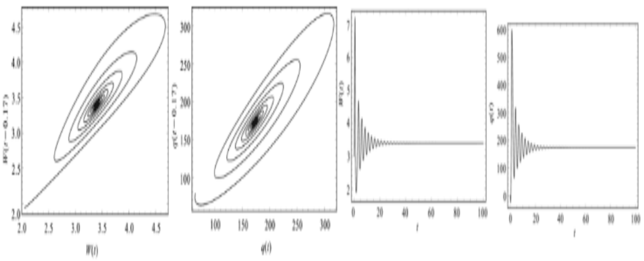


figure 3.Numerical simulation shows the changing process of the system from stable to unstable.

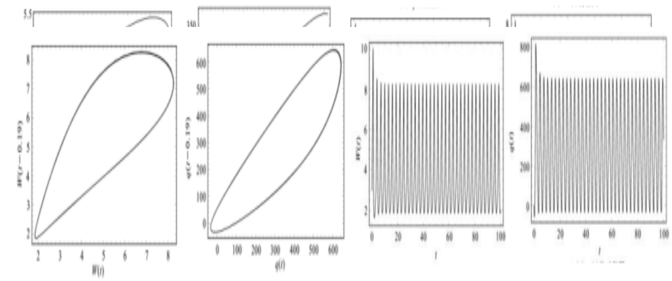
FIG. 1 phase diagram and time history diagram of $W(t)$、$q(t)$ when $\tau = 0.17 < \tau_0$

FIG. 2 phase diagram and time history diagram of $W(t)$、$q(t)$ when $\tau = 0.179008 = \tau_0$

FIG. 3 phase diagram and time history diagram of $W(t)$、$q(t)$ when $\tau = 0.19 > \tau_0$

V. α -STABILITY STUDY

According to literature [7], we have learned the definition of α - stability of dynamical systems with time delay and other



basic knowledge.In this paper, we have also made a brief introduction,we know that α - stability converges faster than stability at the equilibrium point under certain conditions, so it is necessary to study α - stability.In this section, by introducing a α factor, the time-delay system with parameters unrelated to time delay is transformed into time-delay system with parameters related to time delay. Then, the rapid convergence of the fluid flow congestion control model with time delay is analyzed and discussed by using the formula $S_j = \tau - \tau_j$ and its conclusion in the papers of *Beretta* and *Kuang* , that is, the α - stability problem.

In section 3, we obtain the characteristic equation of the fluid flow congestion control model with time delay as equation (9) $\lambda^2 - a_1\lambda - a_2 b_1 e^{-2\lambda\tau} = 0$.

Let $\lambda = s - \alpha$, α is a constant greater than zero, the above equation can be reduced to

$$D(\lambda) = D(s,\tau) = (s-\alpha)^2 - a_1(s-\alpha) - a_2 b_1 e^{-2\tau(s-\alpha)} = 0 \quad (19)$$

Equation (19) can be denoted as $D(s,\tau) = s^2 + cs + d + fe^{-2\tau s} = 0$ (20)

There $c = -a_1 - 2\alpha; d = \alpha^2 + a_1\alpha; f = -a_2 b_1 e^{-2\tau\alpha}$.

When $\tau = 0$, equation (9) is converted to $\lambda^2 - a_1\lambda - a_2 b_1 = 0$, and if it is α -stability,from literature[7], we can get $a_1 < 2\alpha; a_2 b_1 < \alpha^2$.

When $\tau > 0$,let $s = i\omega$,substituted into equation (19) to get

$$-\omega^2 + ic\omega + d + f(\cos 2\omega\tau - i\sin 2\omega\tau) = 0 \quad (21)$$

The above equation separates the real part from the imaginary part,we get:

$$\begin{cases} f\cos 2\omega\tau = \omega^2 - d \\ f\sin 2\omega\tau = c\omega \end{cases} \quad (22)$$

From above,we get $F(\omega,\tau) = \omega^4 + (c-2d)\omega^2 + d^2 - f^2 = 0$

(23)

$$\text{So } \omega^2 = \frac{-c^2 + 2d + \sqrt{c^2 - 4c^2 d + 4f^2}}{2} \quad (24)$$

$$\text{That is } \omega = \sqrt{\frac{-c^2 + 2d + \sqrt{c^2 - 4c^2 d + 4f^2}}{2}} \quad (25)$$

From literature [12] , we get function

$$\sin\theta = \omega\frac{c}{f}, \cos\theta = \frac{\omega^2 - d}{f} \quad (26)$$

$$\text{So } \tan\theta = \frac{\omega c}{\omega^2 - d} \quad (27)$$

From the literature[12] again know $R_n = \tau - \tau_n = \tau - \dfrac{\theta + n2\pi}{2\omega}, n = 0,1,2\cdots\cdots$ (28)

At the same time there is $R(\tau) = \mathrm{sgn}\{\dfrac{d\,\mathrm{Re}(\lambda)}{d\tau}\} = \mathrm{sgn}\{\dfrac{dR_n}{d\tau}\}$

(29)

Let $R_n = 0$, , the zero point is denoted as τ_n , and the derivative of R_n with respect to τ is denoted as $\dfrac{d(R_n)}{d\tau} = 1 > 0$. Obviously, R_n is a monotone increasing function, so R_n has and only has one zero point, non-even number of zero points. According to text [12], the zero point of R_0 is the first and unique stability switching point of model (5), denoted as τ_{01} .

VI. NUMERICAL SIMULATION OF MODEL α - STABILITY

According to the relevant parameters in the numerical simulation of model stability in section 4 and the factor $\alpha = 0.1$, we study the convergence rate of the model at this time, that is α - stability.

We can figure out $c = 2.92072$, $d = -0.302072$, $f = 9.64832$, $\omega = 2.39845$, $\tau_{01} = 0.158143$ by using mathematical software,the model is α - stability.

When $\tau = 0.15 < \tau_{01}$ is selected, the model is α - stability at the equilibrium point, as shown in figure 4;Similarly, when $\tau_{01} < \tau = 0.179 < \tau_0$, the model is asymptotically stable at the equilibrium point, as shown in figure. 5;By comparing figure 4 and figure 5, it is clear that the convergence rate of figure 4 is faster than that of figure 5. Then, with the increase of τ value, the model will gradually become unstable, as shown in figure 3.

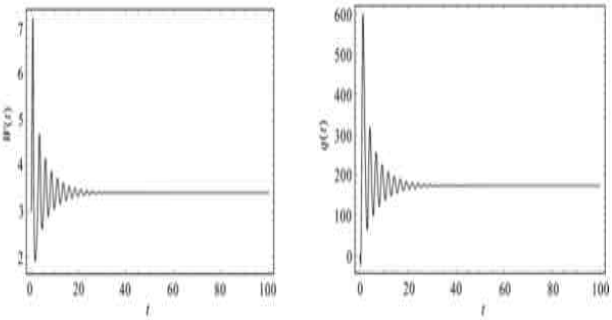


FIG.4 time history diagram of $W(t)$、$q(t)$ when $\tau = 0.15 < \tau_{01}$

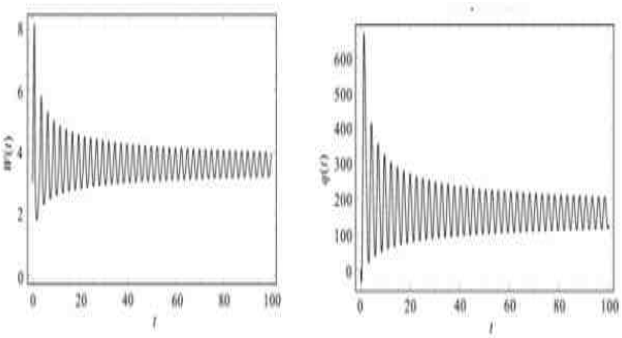


FIG.5 time history diagram of $W(t)$、$q(t)$ when $\tau_{01} < \tau = 0.179 < \tau_0$

CONCLUSION

In this paper, the convergence rate of local stability of an improved fluid flow congestion control model (5) with time delay is studied.Firstly, the stability of model (5) is studied by using the stability geometric switching method,compared with the previous method of constructing central manifolds and using norm, namely *Lyapunov* function method, the method used in this paper is simpler.Therefore, the idea of stability geometric switching has great advantages in the study of local stability of time-delay dynamical systems.Secondly, the α -stability of model (5) is studied by using the time-delay dynamical system stability geometric switching method, that is, by introducing the factor α and controlling its size, the goal of fast convergence can be achieved.At last, the scientific nature of the theory is verified by numerical simulation with mathematical software on the basis of theoretical analysis.

The research results of this paper can be applied to real life. Different parameters are selected for different situations and substituted into the results of this paper to get τ_0 and τ_{01}, so as to carry out detection experiments.The preceding arguments set the stage for further research, and there are many unexplored theories to explore.We can also assign different values to α to obtain different τ_{01}, so as to further study the convergence rate of the model and select the optimal one according to the actual situation, which is conducive to better control of the system.

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