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Fourier series Approximation Method for Solving Nonlinear Optimal Control Problems

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***Abstract*— In this paper we present a numerical method for solving optimal control problems, the optimal pair of control and trajectories of this nonlinear system with quadratic cost functional is obtained by Fourier series approximation. The method is based on expanding time varying functions in the nonlinear system as their Fourier series, by use of the operational matrices for integration and product. The problem is reduced into a set of algebraic equations.**

***Index Terms*— Optimal control, Fourier series approximation, Fourier series operational matrix, Fourier series operational matrix of product**

I. INTRODUCTION

Orthogonal functions are used frequently in various problems of dynamic systems. The main feature of this method is reducing the problems to systems of algebraic equations that can be solved more easily by using programming software like Matlab [8], [9]. Some examples in this area, are using the Walsh functions [22], the block-pulse functions [17], [25], the Chebyshev polynomials [1], [2], [14], the Taylor series [19], [21], the Fourier series [12], [13] and the Legendre polynomials [6], [18].

In this paper control and state functions are approximated by their Fourier series with unknown coefficients that can be find after reducing the nonlinear problem into a set of algebraic equations [15]. The coefficients are calculated in a way that the necessary conditions for minimizing are satisfied [16], [3]. This method can be applied in wild variety of practical problems in other areas [10], [11].

II. PROPERTIES OF FOURIER SERIES

A function over the interval a to b can be expanded into a Fourier series as follows:

$$f(t) = a_0 + \sum_{n=1}^{\infty}\{a_n \cos\left(\frac{2n\pi(t-a)}{b-a}\right) + a_n^* \sin\left(\frac{2n\pi(t-a)}{b-a}\right)\}, \qquad (1)$$

where the Fourier coefficients are found by:

$$a_0 = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t, \qquad (2a)$$

$$a_n = \frac{2}{b-a}\int_a^b f(t)\cos\left(\frac{2n\pi(t-a)}{b-a}\right)\mathrm{d}t, \qquad n = 1,2,3,\ldots, \qquad (2b)$$

$$a_n^* = \frac{2}{b-a}\int_a^b f(t)\sin\left(\frac{2n\pi(t-a)}{b-a}\right)\mathrm{d}t, \qquad n = 1,2,3,\ldots. \qquad (2c)$$

By trauncating the Fourier series up to (2r+1) terms we get:

$$f(t) = a_0 + \sum_{n=1}^{r}\{a_n \cos\left(\frac{2n\pi(t-a)}{b-a}\right) + a_n^* \sin\left(\frac{2n\pi(t-a)}{b-a}\right)\} = A^T\phi(t), \qquad (3)$$

Where

$$A = \left[a_0 \ a_1 \ a_2 \ \ldots \ a_r \ a_1^* \ a_2^* \ \ldots \ a_r^*\right]^T,$$

$$\Phi = \left[\phi_0(t) \ \phi_1(t) \ \ldots \ \phi_r(t) \ \phi_1^*(t) \ \ldots \ \phi_r^*(t)\right]^T,$$

With

$$\phi_n(t) = \cos\left(\frac{2n\pi(t-a)}{b-a}\right), \qquad n = 0,1,2,3,\ldots, \qquad (6a)$$

$$\phi_n^*(t) = \sin\left(\frac{2n\pi(t-a)}{b-a}\right), \qquad n = 1,2,3,\ldots. \qquad (6b)$$

The forward integral of the Fourier series vector Φ(t) can be represented by ([12]):

$$\int_a^t \Phi(t')\,\mathrm{d}t' \cong P\Phi(t), \qquad (10)$$

where P is the Fourier series operational matrix of forward integration and we have:

$$P = (b-a)\begin{bmatrix} \frac{1}{2} & 0 & 0 & \ldots 0 & 0 & \frac{-1}{\pi} & \frac{-1}{2\pi} & \cdots & \frac{-1}{(r-1)\pi} & \frac{-1}{r\pi} \\ 0 & 0 & 0 & \ldots 0 & 0 & \frac{1}{2\pi} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots 0 & 0 & 0 & 0 & \cdots & \frac{1}{2(r-1)\pi} & 0 \\ 0 & 0 & 0 & \ldots 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2r\pi} \\ \frac{1}{2\pi} & \frac{-1}{2\pi} & 0 & \ldots 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{4\pi} & 0 & \frac{-1}{4\pi} & \ldots 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{2r\pi} & 0 & 0 & \ldots 0 & \frac{-1}{2r\pi} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \qquad (11)$$

Moreover we can get ([12]):

$$D = \int_a^b \Phi(t)\Phi^T(t)\,\mathrm{d}t = (b-a)\begin{bmatrix} 1 & & & & 0 \\ & \frac{1}{2} & & & \\ & & \frac{1}{2} & & \\ & & & \frac{1}{2} & \\ & 0 & & \ddots & \\ & & & & \frac{1}{2} \end{bmatrix}. \qquad (12)$$

If we use Eqs.(3) and (10) we get:

$$\int_a^t f(t')\,\mathrm{d}t' = \int_a^t A^T\Phi(t')\,\mathrm{d}t' = A^T P\Phi(t). \qquad (14)$$

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III. OPTIMAL CONTROL PROBLEMS

Optimal control theory deals with finding a control law for a dynamical system over a period of time such that an objective function is optimized. For example, the dynamical system might be a spacecraft with controls corresponding to rocket thrusters, and the objective might be to reach the moon with minimum fuel expenditure [4], [5].

In this section we solve an optimal control problem by using Fourier series approximation method described above.
Here we need to minimize a quadratic cost function with respect to some constraints and initial and boundary conditions are given.
First we write each unknown function in the nonlinear optimal control problem as their Fourier series approximation then we substitute the new functions into the given problem, then by using Fourier series properties explained before we will get a system of nonlinear algebraic equations as our constraints and we will have a nonlinear function as our cost function. Then we can solve it by using common programming software.

IV. NUMERICAL METHOD AND EXAMPLES

Consider the following optimal control problem [7]:

$$\text{Min } J = \int_0^1 0.5u(t)^2$$

Subject to:

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = u(t)$$

With these initial and boundary conditions:

$$x_1(0) = 1, x_1(1) = 0$$
$$x_2(0) = 1, x_2(1) = 0$$

Now we have three unknown functions that write them as their Fourier series approximation:

$$x_1(t) = A^T \phi(t)$$
$$x_2(t) = B^T \phi(t)$$
$$u(t) = C^T \phi(t)$$

Where A, B and C are Fourier series coefficients and are unknown.

Now if we use these three equations in our cost function, we get:

$$J = \int_0^1 0.5u^2(t) = \int_0^1 0.5C^T \phi(t) C^T \phi(t)$$

$$= \int_0^1 0.5C^T \phi(t)\phi(t) C$$

$$= 0.5C^T DC$$

Now by applying the Fouries series in constraints and integration from 0 to t one can get:

$$\int_0^t x_1'(t) = \int_0^t x_2(t)$$

$$x_1(t) - x_1(0) = \int_0^t B^T \phi(t) dt,$$

$$A\phi(t) - A_0\phi(t) = B^T P\phi(t).$$

$$\int_0^t x_2'(t) = \int_0^t u(t) = \int_0^t C\phi(t) dt$$

$$x_2(t) - x_1(0) = CP\phi(t)$$

$$B\phi(t) - B_0\phi(t) = CP\phi(t).$$

Where

$A_0 = [1, 0, ..., 0, ..., 0]$, B0=$[1, 0, ..., 0, ..., 0]$

By eliminating $\varphi(t)$ from both sides of above equations we have:

$$\text{Min } J = 0.5C^T DC$$

Subject to:

$$A^T - A_0 - B^T P = 0$$
$$B^T - B_0 - C^T P = 0$$

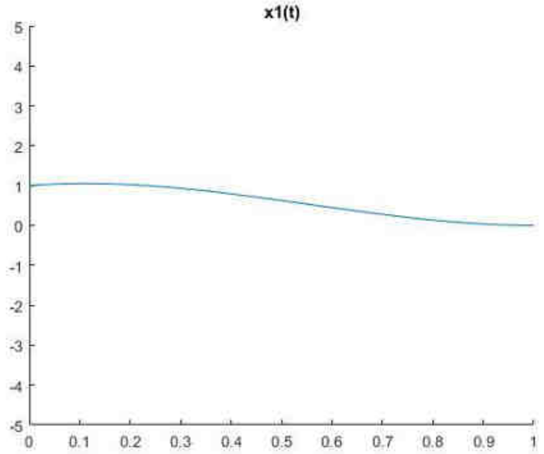
For boundary and initial conditions, we get:

$$A\phi(0) = 1, \quad B\phi(0) = 1,$$
$$A\phi(1) = 0, \quad B\phi(1) = 0.$$

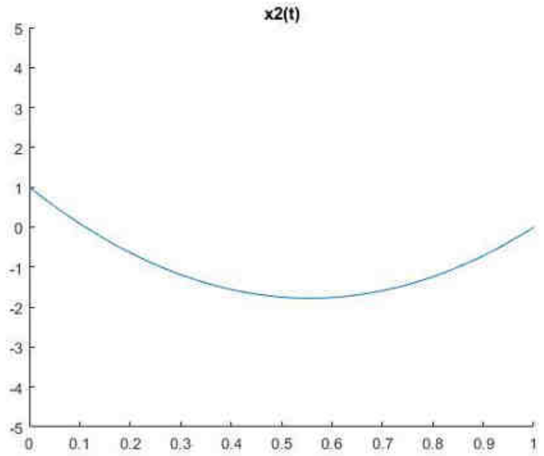
Therefore, we get a nonlinear cost function and a system of nonlinear equations as our constraints that can be solved using Matlab software. The results are shown in next section.

V. FIGURES

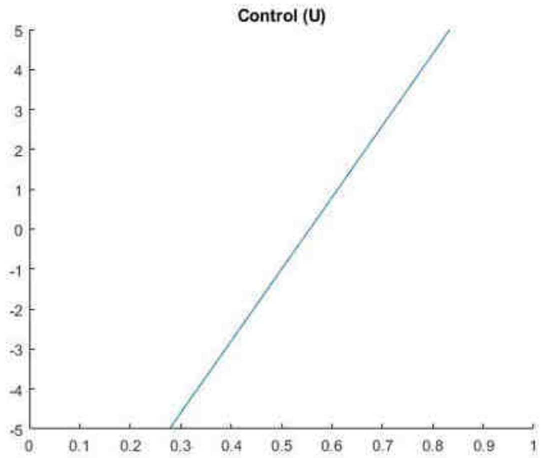
Graph of x1(t):



Graph of x2(t):



Graph of u(t):



VI. CONCLUSION

In this paper Fourier series were used for approximating unknown functions in the optimal control problem, then this method reduced the problem into a set of nonlinear algebraic equations that can be solved using many numerical methods. The advantage of this method is for solving optimal control problems that finding their exact solutions is so difficult, so applying this method, help us to approximate solutions.

By using operational matrices, the nonlinear optimal control model is reduced into a set of nonlinear algebraic equations. Since the operational matrices of integration

and product contain many zero elements, its makes the method computationally attractive.

One example shows the effectiveness of the method.

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