Numerical Solution Exploration of Stochastic Dynamical System Based on Mathematica

Dongwei Huang, Lijiao Cai

***Abstract*—In recent years, stochastic differential equations have been widely used in financial economy, engineering technology, biological mathematics and other fields. Based on Mathematica system, this paper improves the existing numerical simulation methods and obtains a new numerical simulation algorithm. First, we give the derivation of the original Milstein numerical simulation algorithm. Secondly, we apply the idea of Runge-Coulthard method to the existing Milstein method to obtain a new numerical simulation algorithm. Then, for two common bioeconomic dynamic systems, a more realistic stochastic dynamic system is established by considering the influence of non-negligible noise in real life. This paper not only analyzes the dynamic behavior of the state variable, but also applies the improved simulation method to the system. The numerical simulation results verify our theoretical analysis, and then show that this method is feasible.**

***Index Terms*—Mathematica,Milstein stochastic differential method, Stochastic biological mathematical model,Stochastic economic dynamics model.**

I. INTRODUCTION

In 1987,the publication of "Principia Mathematica of Natural Philosophy" marked the first appearance of the important concept of "differential equations" in mathematics and physics.Next,to characterize natural phenomena in physics,Newton established deterministic differential equations,known as ordinary differential equations (ODE).Since then,ODE have been widely used in natural science,engineering and other fields,which greatly accelerating the pace of social development.

It is well known that when we use ordinary differential equations to describe the change of state variables over time in real life,the common simplification is to ignore some minor factors.For example,when describing actual physical phenomena,it is usually to build mathematical models to describe them.However,this kind of mathematical model is usually an ideal mathematical model after some random factors are ignored. However, in signal processing,finance and other fields, this random factor is often the key to the problem,so it can not be ignored.The stochastic differential equation (SDE) is a subject combining certainty and uncertainty, so it can more accurately describe the objective laws of many phenomena in nature [1].Therefore,the research on SDE is of great practical significance.

In 1951,the Japanese mathematician Ito in "On Stochastic Differential Equations" put forward the Ito integral,which laid the foundation of SDE theory research[2].Driven by a large number of practical problems,the research on SDE has made rapid development. In recent years,SDE has been more and more widely used in the fields of financial economy,engineering technology, ecology and so on, so the research on equation solutions becomes very important. However,due to the uncertainty of SDE, the solutions of most equations are generally unavailable or difficult to be obtained directly, so numerical solutions play an important role in the study of SDE [3].

Common numerical methods of SDE include Eular method[4],Milstein method[5],Roung-Kutta method etc. However,the accuracy of the Eular method is not high enough to meet the needs of practical applications. This paper mainly considers Ito SDE. We first derived the Milstein method,and then applied the idea of Roung-Kutta method in the ODE to the Milstein method to improve the existing Milstein method. Then we implement the above process in Mathematica software system and give the corresponding program. Secondly, for two kinds of common deterministic dynamical systems, by considering the influence of noises that cannot be ignored in real life,we establish a more practical stochastic dynamical system and study the behavior of its solutions. Finally,the improved algorithm is applied to these two systems, moreover, our analysis is verified by numerical simulation results.

II. ALGORITHM FORMULATION

Let N be a positive integer,

$$\Delta t = \frac{T - t_0}{N}, t_n = t_0 + n\Delta t, n = 0,1\ldots ,$$

$$\Delta W_n = W(t_n + \Delta t) - W(t_n) \sim \sqrt{\Delta t}B, B \sim N(0,1),$$

Let $X(t) = X_t, W(t) = W_t$ .First, consider the stochastic differential equation with solution X_t :

$$dX_t = f(X_t)dt + g(X_t)dW_t \tag{1}$$

Then, the integral form of equation (1) is:

$$X_t = X_{t_0} + \int_{t_0}^{t} f(X_s)dt + \int_{t_0}^{t} g(X_s)dW_s \tag{2}$$

The second term is Ito integral, $t \in [t_0,T]$, $f(X_s)$, $g(X_s)$ are continuously measurable functions which satisfying the linear growth condition on $[t_0,t]$.

For any quadratic differentiable function $V: R \to R$, Ito formula is used to obtain[6]:

$$dV(X) = \left(f(X)\frac{\partial V(X)}{\partial X} + \frac{1}{2}g(X)^2\frac{\partial^2 V(X)}{\partial X^2}\right)dt + g(X)\frac{\partial V(X)}{\partial X}dW \tag{3}$$

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Write it as an integral:

$$V(X_t) = V(X_{t0}) + \int_{t0}^{t}\left[f(X_s)\frac{\partial V(X_s)}{\partial X_s} + \frac{1}{2}g(X_s)^2\frac{\partial^2 V(X_s)}{\partial (X_s)^2}\right]ds + \int_{t0}^{t} g(X_s)\frac{\partial V(X_s)}{\partial X_s}dW = V(X_{t0}) + \int_{t0}^{t} L^0 X(X_s)ds + \int_{t0}^{t} L^1 X(X_s)dW_s \tag{4}$$

Among them $t \in [t_0, T]$

$$L^0 = f(X_s)\frac{\partial}{\partial X_s} + \frac{1}{2}g(X_s)^2\frac{\partial^2 V(X_s)}{\partial (X_s)},\quad L^1 = g(X_s)\frac{\partial}{\partial X_s}.$$

Let $V = f$ and $V = g$ [7], respectively, and apply Ito formula (4) to (2):

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} V(X_s)ds + \int_{t_0}^{t} V(X_s)dW_s \\ &= X_{t_0} + f(X_{t_0})\int_{t_0}^{t} ds + g(X_{t_0})\int_{t_0}^{t} dW_s + \int_{t_0}^{t}\int_{t_0}^{t} L^0 f(X_p)dpds \\ &= \int_{t_0}^{t}\int_{t_0}^{t} L^1 f(X_p)dW_p ds + \int_{t_0}^{t}\int_{t_0}^{t} L^0 g(X_p)dpdW_s \\ &\quad + \int_{t_s}^{t}\int_{t_s}^{t} L^1 g(X_p)dW_p dW_s \end{aligned} \tag{5}$$

Continue to use Ito formula for equation (5), let $V = L^1 g$, then

$$\begin{aligned} X_t = X_{t0} &+ f(X_{t0})\int_{t0}^{t} ds + g(X_{t0})\int_{t0}^{t} dW_s + \int_{t0}^{t}\int_{t0}^{s} L^1 g(X_{t0})dW_p dW_s \\ &+ \int_{t0}^{t}\int_{t0}^{s} L^0 g(X_p)dpdW_s + \int_{t0}^{t}\int_{t0}^{s} L^0 f(X_p)dpds \\ &+ \int_{t0}^{t}\int_{t0}^{s} L^1 f(X_p)dW_p dW_s + \int_{t0}^{t}\int_{t0}^{s}\int_{t0}^{p} L^0 L^1 g(X_q)dqdW_p dW_s \\ &+ \int_{t0}^{t}\int_{t0}^{s}\int_{t0}^{p} L^1 L^1 g(X_q)dW_q dW_p dW_s \end{aligned} \tag{6}$$

Due to $dW_t = 0dt + 1dW_t$, let $V = \frac{1}{2}W_t^2$. From formula Ito, we get

$$dV = d\left(\frac{1}{2}W_t^2\right) = \frac{1}{2}dt + W_t dt \tag{7}$$

Writing it as an integral:

$$\int_0^t \frac{1}{2}W_s^2 ds = \int_0^t \frac{1}{2}ds + \int_0^t W_s dW_s \tag{8}$$

So

$$\int_0^t W_s dW_s = \int_0^t \frac{1}{2}W_s^2 ds - \int_0^t \frac{1}{2}ds = \frac{1}{2}(W_t^2 - t) \tag{9}$$

Then

$$I_{11} = \int_{t_n}^{t_{n+1}} I_1(s)dW_s = \frac{1}{2}[(W_{n+1} - W_n)^2 - \Delta t] \tag{10}$$

We can get:

$$X_{n+1} = X_n + I_0 f(X_n) + I_1 g(X_n) + I_{11}L^1 g(X_n) + R \tag{11}$$

Where R is the high-order residual term, $I_0 = \Delta t$, $I_1 = \Delta W_n$. The random differential method can be obtained by truncate-ing at appropriate positions in the random a expansion. The general form is:

$$X_{n+1} = X_n + f(X_n)\Delta t + g(X_n)\Delta W_n + \frac{1}{2}[(W_{n+1} - W_n)^2 - \Delta t]g(X_n)\frac{\partial g(X_n)}{\partial X_n} \tag{12}$$

Based on the original Milstein method, we apply the Runge-Kutta method in the numerical solution of ordinary differential equations to this method, and improve the existing numerical solution.

$$f_1 = f(X_n); f_2 = f(X_n + \frac{1}{2}f_1); f_3 = f(X_n + \frac{1}{2}f_2); f_4 = f(X_n + f_3) \tag{13}$$

$$X_{n+1} = X_n + \frac{\Delta t}{6}(f_1 + 2f_2 + 2f_3 + f_4) + g(X_n)\Delta W_n + \frac{1}{2}[(W_{n+1} - W_n)^2 - \Delta t]g(X_n)\frac{\partial g(X_n)}{\partial X_n} \tag{14}$$

III. Model analysis and numerical simulation

A. Predator-prey stochastic dynamical system with Holling functional response

With the development of science and technology, ecological problems have gradually become a problem closely related to human life. In ecology, population ecology is the most systematic and mature branch. It is a subject that describes the dynamic relationships among populations, environments, and interactions among populations[8]. Common relationships between populations and environment and between populations include competition, prey-predator, and mutualism.

In order to describe these relationships more accurately, biologist Holling proposed different functional response functions for different biological populations in 1965. These predator-prey models with different Holling functional responses can be expressed as:

$$\begin{cases} dx(t) = [x(t)f(x(t)) - y(t)p(x(t))]dt \\ dy(t) = [ky(t)p(x(t)) - y(t)g(y(t))]dt \end{cases} \tag{15}$$

Where *x(t)* represents the prey population, *y(t)* represents the predator population, *p(x(t))* represents the predator functional response function, *g(y(t))* represents the predator mortality without predation, and *f(x(t))* represents the prey population without predation.

In [9], a predator-prey model is proposed as follows:

$$\begin{cases} dx(t) = x(t)(1 - y(t))dt \\ dy(t) = y(t)(A + By(t) + Cy(t)^2 + x(t))dt \end{cases} \tag{16}$$

Where A, B, C is constant. In this paper, by making the transformation, it is found that when $B + 2C > 0$ and $B^2 - 4AC < 0$, if there is a positive number k satisfying $(B+2)^2 < 3C(A+2k)$, system (16) has a limit cycle in R_+^2 and it is unique. When $B + 2C < 0$, system (16) has no limitcycle in R_+^2. However, this model does not take into account the impact of random factors in real life, such as earthquakes and natural disasters, which cannot be ignored for the development of population. Therefore, we improve the system (16) and obtaine the following predator-prey model that is closer to the reality:

$$\begin{cases} dx(t) = x(t)(1 - y(t))dt + \sigma_1 x(t)dW_1(t) \\ dy(t) = y(t)(A + By(t) + Cy(t)^2 + x(t))dt + \sigma_2 y(t)dW_2(t) \end{cases} \tag{17}$$

Where σ_1, σ_2 are constants. Wn(t) is the independent standard Brownian motion.

Definition 3.1. The solution of model (17) is said to be random and finally bounded. If there is a positive number C for $\forall \varepsilon \in (0,1)$, and the solution of model (17) satisfies the following criteria for any initial value $(x_0,y_0) \in R_+^2$:

$$\lim_{t\to\infty}\sup P\{|(x(t),y(t))| = \sqrt{x(t)^2+y(t)^2} > C\} < \varepsilon \tag{18}$$

Theorem 3.1. The solution of model (17) is random and ultimately bounded.

Proof:So let's define:

$$V(x,y) = x^q + y^q,\ (x,y) \in R_+^2 \tag{19}$$

From formula Ito:

$$dV(x,y) = LV\,dt + \sigma_1 q x^q dW_1 + \sigma_2 q y^q dW_2 \tag{20}$$

Where $LV: R^2 \to R$ is defined as:

$$LV = qx^q(1-y) + qy^q(A+By+Cy^2+x) + \frac{1}{2}\sigma_1^2 q(q-1)x^q + \frac{1}{2}\sigma_2^2 q(q-1)y^q \tag{21}$$

From $0 < q < 1$ we get:

$$LV \le qx^q(1-y) + qy^q(A+By+Cy^2+x) \le K - V(x,y) \tag{22}$$

Where K is an integer.Substitute inequality (22) into inequality (20):

$$dV(x,y) \le (K - V(x,y))dt + (\sigma_1 q x^q dW_1 + \sigma_2 q y^q dW_2) \tag{23}$$

$$\begin{aligned} d(e^t V(x,y)) &= e^t[V(x,y)dt + dV(x,y)] \\ &\le Ke^t dt + e^t \sigma_1 q x^q dW_1 + e^t \sigma_2 q y^q dW_2 \end{aligned} \tag{24}$$

$$e^t EV(x(t),y(t)) \le V(x_0,y_0) + K(e^t - 1) \tag{25}$$

So

$$\lim_{t\to\infty}\sup EV(x(t),y(t)) \le K \tag{26}$$

On the other hand,

$$|(x,y)|^q = (x^2+y^2)^{\frac{q}{2}} \le (2xy)^{\frac{q}{2}} \le 2^{\frac{q}{2}} \max\{x^q,y^q\} \le 2^{\frac{q}{2}} V(x,y) \tag{27}$$

Then

$$\lim_{t\to\infty}\sup E|x(t),y(t)|^q \le 2^{\frac{q}{2}} \lim_{t\to\infty}\sup EV(x(t),y(t)) \le 2^{\frac{q}{2}} K \tag{28}$$

To make $q = \frac{1}{2}, \exists K_1 > 0$

$$\lim_{t\to\infty}\sup E|(x(t),y(t))|^{\frac{1}{2}} \le K_1 \tag{29}$$

For $\forall \varepsilon > 0$, let's take $C = \frac{K_1^2}{\varepsilon^2}$, and we get that from inequality Chebyshev:

$$\lim_{t\to\infty}\sup P\{|(x(t),y(t))| > C\} \le \varepsilon \tag{30}$$

Therefore, the solution of model (17) is finally bounded.

The algorithm proposed in this paper is used to conduct numerical simulation of the model (17), and the following discrete form is obtained from the equation (14):

$$\begin{aligned}
f_1 &= x(t_k)(1 - y(t_k)); \\
f_2 &= (x(t_k)+\tfrac{1}{2}f_1)[1-(y(t_k)+\tfrac{1}{2}g_1)]; \\
f_3 &= (x(t_k)+\tfrac{1}{2}f_2)[1-(y(t_k)+\tfrac{1}{2}g_2)]; \\
f_4 &= (x(t_k)+f_3)[1-(y(t_k)+g_3)]; \\
g_1 &= y(t_k)[A+By(t_k)+Cy(t_k)^2+x(t_k)]; \\
g_2 &= (y(t_k)+\tfrac{1}{2}g_1)[A+B(y(t_k)+\tfrac{1}{2}g_1)+C(y(t_k)+\tfrac{1}{2}g_1)^2+(x(t_k)+\tfrac{1}{2}f_1)]; \\
g_3 &= (y(t_k)+\tfrac{1}{2}g_2)[A+B(y(t_k)+\tfrac{1}{2}g_2)+C(y(t_k)+\tfrac{1}{2}g_2)^2+(x(t_k)+\tfrac{1}{2}f_2)]; \\
g_4 &= (y(t_k)+g_3)[A+B(y(t_k)+g_3)+C(y(t_k)+g_3)^2+(x(t_k)+f_3)]
\end{aligned} \tag{31}$$

$$\begin{cases} x_{k+1} = x_k + \dfrac{\Delta t}{6}(f_1+2f_2+2f_3+f_4) + \sigma_1 x_k \Delta W_1 + \dfrac{1}{2}\sigma_1^2 x_k(\Delta W_1^2 - \Delta t) \\ y_{k+1} = y_k + \dfrac{\Delta t}{6}(g_1+2g_2+2g_3+g_4) + \sigma_2 y_k \Delta W_2 + \dfrac{1}{2}\sigma_2^2 y_k(\Delta W_2^2 - \Delta t) \end{cases} \tag{32}$$

The results are shown in the figures below. Figure (1-6) shows the phase diagram of the system (17) under different parameters.

Figure 1.
A = 10, B = 0.8, C = 0.3.

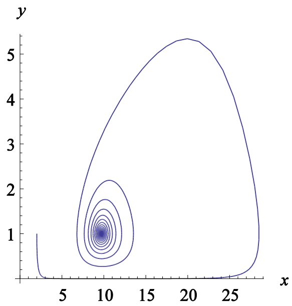


Figure 2.
A = −10, B = 0.8, C = −0.5.

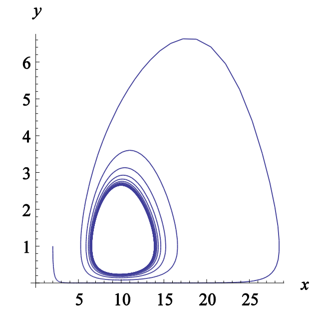


Figure 3.
A = 10, B = 0.8, C = 0.3,
$\sigma_1 = 0.1,\ \sigma_2 = 0.1.$

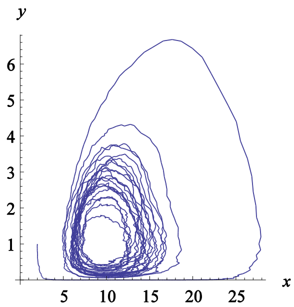


Figure 4.
A = 10, B = 0.8, C = 0.3,
$\sigma_1 = 0.1,\ \sigma_2 = 0.3.$

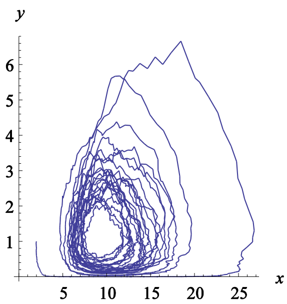


Figure 5:
A = 10, B = 0.8, C = 0.3,
$\sigma_1 = 0.3,\ \sigma_2 = 0.1.$

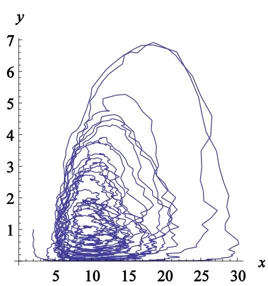
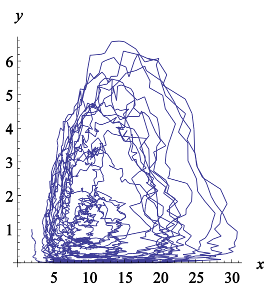


Figure 6:
A = 10, B = 0.8, C = 0.3,
$\sigma_1 = 0.3,\ \sigma_2 = 0.3.$



From figures (1-2), we can see that when $B+2C > 0$ and

$B^2 - 4AC < 0$, take $0 < k < 0.64$, the system (16) exists a limit cycle and is unique. When $B + 2C < 0$, there is no limit cycle for the system (16). As can be seen from figure (3-6), when the random disturbance to the two populations is small, the number of predator and prey populations does not change much, and the shape of limit cycle basically exists. And with the increase of random disturbance intensity, the prey population is more affected than the predator population. After several experiments, we found that when the intensity increased to about 0.3, the periodic oscillations formed by the two populations with the change of time ceased to exist.

IV. A kind of random financial system

Now, China is in an era of rapid development. As an important indicator of social development, the security, stability and order of the economic system is the key to the stable development of the whole society[11]. Therefore, it is of great significance to study it both academically and practically. Traditional economic theory holds that the equilibrium is stable when the general characteristics of the economic system[12]. But with the rapid development of the society, the traditional linear economic theory is unable to explain the complicated economic phenomenon. Thanks to the joint efforts of many economists, we have come to realize that nonlinearity is the essential attribute of economic system. Driven by the current economy, nonlinear economics has gradually become the frontier field of economics research.

In [13], the following nonlinear economic dynamics model is proposed:

$$\begin{cases} dX = (Z + (Y - \alpha)X)dt \\ dY = (1 + \beta Y - X^2)dt \\ dZ = (-X - \gamma Z)dt \end{cases} \tag{33}$$

Where X is the interest rate, Y is the investment demand, Z is the price index demand, α is the amount of savings, β is the cost of each investment, and γ is the price elasticity of demand.

By using the nonlinear theory, they obtain if $\gamma - \beta - \alpha\beta\gamma < 0$, $\gamma + \alpha - \frac{1}{\beta} > 0$, then the equilibrium point P of system (23) is a stable sink. If $\gamma - \beta - \alpha\beta\gamma < 0$, $\gamma + \alpha - \frac{1}{\beta} < 0$, point P is the saddle point. If $\gamma - \beta - \alpha\beta\gamma < 0$, $\gamma + \alpha - \frac{1}{\beta} = 0$, then Hopf bifurcation occurs at point P and a family of periodic solutions exists. If $\gamma - \beta - \alpha\beta\gamma = 0$, $0 < \gamma < 1$, then point P is the nonhyperbolic unstable equilibrium point. If $\gamma - \beta - \alpha\beta\gamma = 0$, $\gamma > 1$, then point P bifurcates[13-14]. However, this paper does not take into account such random factors as government macroeconomic decision-making intervention, unexpected political events, outbreak of war, natural disasters and other random factors in the economic system. Therefore, based on the previous studies of scholars, we improve the above model and obtaine the following stochastic economic dynamic system model:

$$\begin{cases} dX = (Z + (Y - \alpha)X)dt + \sigma_1 X\,dW_1 \\ dY = (1 + \beta Y - X^2)dt + \sigma_2 Y\,dW_2 \\ dZ = (-X - \gamma Z)dt + \sigma_3 Z\,dW_3 \end{cases} \tag{34}$$

$\sigma_1, \sigma_2, \sigma_3$ are constants, W_1, W_2, W_3 are the wiener process with a mean of 0 and a variance of Δt.

The improved Milstein method proposed in this paper is used to simulate the model (34). Firstly, the discretization format of model (34) is obtained from (14):

$$f_1 = z_k + (y_k - \alpha)x_k;\ g_1 = 1 + \beta y_k - x_k^2;\ h_1 = -x_k - \gamma z_k;$$

$$f_2 = (z_k + \tfrac{1}{2}h_1) + [(y_k + \tfrac{1}{2}g_1) - \alpha](x_k + \tfrac{1}{2}f_1);$$

$$g_2 = 1 + \beta(y_k + \tfrac{1}{2}g_1) - (x_k + \tfrac{1}{2}f_1)^2;\ h_2 = -(x_k + \tfrac{1}{2}f_1) - \gamma(z_k + \tfrac{1}{2}h_1);$$

$$f_3 = (z_k + \tfrac{1}{2}h_2) + [(y_k + \tfrac{1}{2}g_2) - \alpha](x_k + \tfrac{1}{2}f_2);$$

$$g_3 = 1 + \beta(y_k + \tfrac{1}{2}g_2) - (x_k + \tfrac{1}{2}f_2)^2;\ h_3 = -(x_k + \tfrac{1}{2}f_2) - \gamma(z_k + \tfrac{1}{2}h_2);$$

$$f_4 = (z_k + h_3) + [(y_k + g_3) - \alpha](x_k + f_3);$$

$$g_4 = 1 + \beta(y_k + g_3) - (x_k + f_3)^2;\ h_4 = -(x_k + f_3) - \gamma(z_k + h_3); \tag{35}$$

$$\begin{cases} X(t_{k+1}) = X(t_k) + \frac{\Delta t}{6}(f_1 + 2f_2 + 2f_3 + f_4) + \sigma_1 X(t_k)dW_1 + \frac{1}{2}\sigma_1^2 X(t_k)(dW_1^2 - \Delta t) \\ Y(t_{k+1}) = Y(t_k) + \frac{\Delta t}{6}(g_1 + 2g_2 + 2g_3 + g_4) + \sigma_2 Y(t_k)dW_2 + \frac{1}{2}\sigma_2^2 Y(t_k)(dW_2^2 - \Delta t) \\ Z(t_{k+1}) = Z(t_k) + \frac{\Delta t}{6}(h_1 + 2h_2 + 2h_3 + h_4) + \sigma_3 Z(t_k)dW_3 + \frac{1}{2}\sigma_3^2 Z(t_k)(dW_3^2 - \Delta t) \end{cases} \tag{36}$$

The simulation results are as follows.

Figure 7: Time chart of system (34) with $\alpha = 3.5, \beta = 0.2, \gamma = \frac{3}{35}, \sigma_1 = \sigma_2 = \sigma_3 = 0.2$.

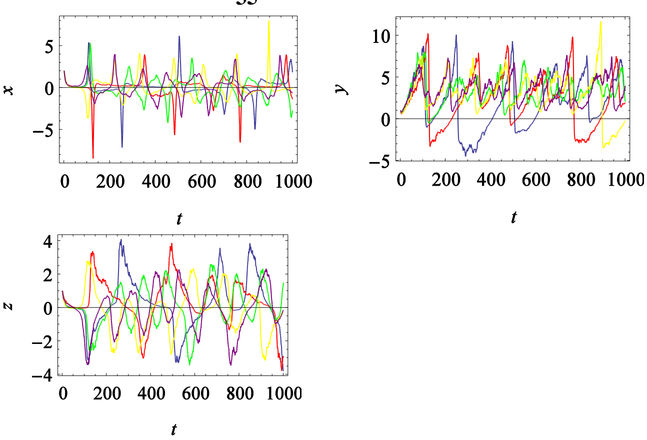


Figure 8.

$a = 3.5, b = 0.2, g = \frac{3}{35}$.

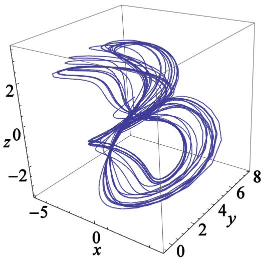


Figure 9.

$a = 3.5, b = 0.2, g = \frac{3}{35}$

$s_1 = s_2 = s_3 = 0.2$

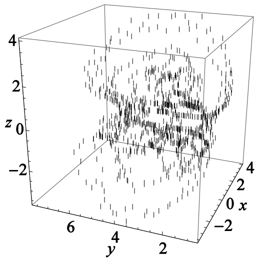


Figure 10.

$a = 3.5, b = 0.2, g = \frac{3}{35}$,

$s_1 = s_2 = s_3 = 0.4$.

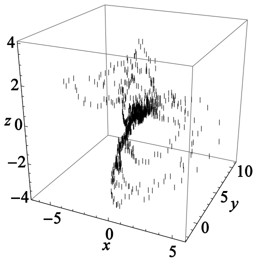
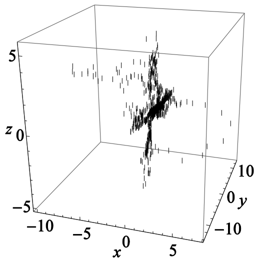


Figure 11.

$a = 3.5, b = 0.2, g = \frac{3}{35}$,

$s_1 = s_2 = s_3 = 0.6$.



From figure (7-11), by comparing the stochastic system with the corresponding deterministic system, it is found that the existence of the stochastic term does change the characteristics of the deterministic system, and the system becomes more chaotic with the increase of the random disturbance intensity. In a complex financial system, interest rate, investment demand and price index demand are interdependent and interrelated. We can adjust the development of the market and increase the vitality of the market by changing the institutional parameters. When the market is stagnant or rigid, we can also adjust the parameters to make the market healthy.

V. Conclusion

This paper mainly improves the existing Milstein method based on Mathematica. Secondly, two common deterministic bioeconomic dynamics models are improved, and a more realistic stochastic bioeconomic dynamics model is proposed. Then we applied the text algorithm to these two types of models.Through numerical simulation, we found that the existence of random disturbance factors that could not be ignored in life would indeed have an impact on the system state variables. Moreover, when the random disturbance intensity increased to a certain degree, the system would completely collapse and the value of the state variable would approach zero. This means that through our numerical simulation, the state of the system can be predicted in advance and the random disturbance tolerance of the system can be evaluated, thus providing theoretical guidance for us to take the next step, which has very important theoretical research value and practical application value.

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