Preparation Consensus Control of Multi-Agent Systems with Time Delay Based On Second-Order Bessel-Legendre

Jing Zhang

Abstract— **For the conservatism of second-order multi-agent systems with time delay, there is still room for improvement of the Wirtinger-based inequality. In the process of stability analysis of multi-agent systems with time delay, the introduction of the second-order Bessel-Legendre inequality to amplify the integral term is smaller than the Wirtinger- based inequality. The second order Bessel-Legendre inequa- lity is closer to the true value of the integral term. In addition, in order to derive less conservative stability conditions, a triple integral term is added when constructing the Lyapunov functional. However, Wirtinger-based inequality cannot direc- tly deal with the double integral term generated by the derivation of the triple integral. Therefore, the Wirtinger- based double integral inequality is introduced to solve this problem. Based on Lyapunov stability theorem and linear matrix inequality theory, the stability conditions that ensure the multi-agent agreement and less conservative are obtained.**

Index Terms— **Multi-agent systems, Consensus, Time-varying delays, Second order Bessel-Legendre inequality**

I. INTRODUCTION

During the last decade, cooperative control of multi- agent systems has gained much attention from many fields, such as biology, physics, computer science and so on [1–4]. Consensus is a fundamental problem achieved by sharing information among the neighborhood. For a long time, researchers have focused their research on the first- order multi-agent system [5]-[10], but due to the simple structure of the first-order multi-agent system, in the real world, the actual reality cannot be accurately described. For relatively complex systems, considering that second- order dynamics can be used to model more complex processes in reality, people began to shift the focus of research from first-order multi-agent systems to second- order multi-agent systems. Ren et al. [11] show that the conditions that can make the first-order multi-agent system reach agreement are not app- licable to the second-order system. At the same time, they proposed the original second-order multi-agent consensus control protocol. On this basis, the research results of the consensus of second-order multi-agent systems have grad- ually appeared in various fields. Literature [12] discusses the second-order multi-agent system based on the event-trigge- red control protocol. Literature [13]-[15], considers the sec- ond-order multi-agent system with sampled position data. Literature [16]-[18],uses different control algorithms solve the second-order consensus problem of linear multi-agent system and nonlinear multi-agent system in finite time , respectively. Further, the literature [19]-[20], carried out research on the consensus of discrete-time second-order systems in different situations, and proposed conditions for ensuring that multi-agents reach agreement.

In real life, due to the limited ability of each UAV to process information and the adverse effects of external interference, there is a general time delay in the process of information interaction [21]-[22]. Therefore, the research on the consensus of multi-agent systems with time delay systems has received widespread attention. In the research process of multi-agent systems with time delay, there are mainly two ways to reduce the conservativeness of the system. The first way is to add a new polynomial when constructing the Lyapunov function. Wang et al. [23] construct a new Lyapunov function by introducing a triple integral term, which solves the problem of the first-order delay in the directed topology. The consensus problem of the multi-agent system and its effectiveness in reducing the conservativeness of the system are verified. Another method is to find an integral inequality closer to the true value to scale the integral term in the derivative when dealing with the derivative of the Lyapunov function. Ao et al.[24] and Yu et al. [25], based on Jensen' s inequality, respectively discussed the multi-agent systems with undirected topology and the second-order leader-following multi-agent system with and without time delay under external disturbance. However, the stability criterion obtained based on the Jensen integral inequality is relatively conservative. Therefore, in order to further reduce the conservativeness of the system, the literature [26] uses Wirtinger-based inequality to derive the consensus convergence of a second-order multi-agent network with non-uniform time delay. Furthermore, literature [27] constructed a new Lyapunov functional by introducing integral terms and new augmented vectors, and used the extended relaxation integral inequality combining Wirtinger integral inequality and convex combination method to deal with the derivative of the functional. The consensus problem of the first-order delay multi-agent system with the directed topology is solved, and the stability condition with less conservativeness is obtained.

Inspired by some previous works ,aiming at the conservative problem of the second-order multi-agent systems with time delay, there is still room for improvement based on Wirtinger inequality. In [28], Liu et al. integrated the characteristics of the second-order Bessel-Legendre inequality into the construction of the Lyapunov function. This inequality enlarges the integral term to a lesser degree

Manuscript received August 12, 2021
Jing Zhang, School of Electronic and Information Engineering, Tianjin Vocational Institute, Tianjin, China

than Wirtinger's inequality, and is closer to the true value of the integral term.In order to obtain lower conservatism, a new Lyapunov function with triple integral form can be constructed. However, Wirtinger-based inequalities cannot directly deal with the double integral terms generated by the triple integral derivation. Therefore, based on Wirtinger inequality, literature [29] introduced a double integral inequality based on Wirtinger and derives the stability criterion related to time delay, which verified the validity and practicability of the inequality. But the literature [29] did not consider the case of second-order delay multi-agent systems.

The remaining of this paper is organized as follows.In Section 2 some graph theory,model formulations and preliminaries are given.Main results are provided in Section 3.In Section 4 a numerical examples is designed to verify our results.Section 5 draw the conclusions.

II. PRELIMINARIES AND MODEL FORMULATION

A. Graph Theory Notations

The considered MASs is consisted by n agents.In order to keep generality,the n agents are labeled by node 1, 2, . . ., n.For a directed $G=(\mathrm{V},\mathrm{E},\mathrm{A})$,where $\mathrm{E}\subseteq \mathrm{V}\times\mathrm{V}$ and $\mathrm{V}=\{v_1,\ v_2,\ \ldots,\ v_n\}$ represent edge set and node set,respectively.Node i stands for ith agent,and the directed edge of G is an order node pair (v_{i} , v_{j}) standing for the direction of information flow from j to i,but direction of information flow can not from i to j. However ,the undirected graph is opposite to directed graph.Its (v_{i} , v_{j}) represents the direction of information flow is mutual.Thus, the undirected graph can be regarded as a special directed graph $\mathrm{A}=\left[a_{ij}\right]\in R^{n\times n}$ represents a whighted adjacency matrix and its elements are non-negative.It can be said that G has a directed spanning tree ,when least a node has least a directed path to any others.

B. Time-delay Multi-agent System Description

Consider the second-order multi-agent system with time delay ,and its dynamics described as:

$$\dot{x}_i=\upsilon_i \quad \dot{\upsilon}_i=u_i \tag{1}$$

where x_i , v_i , u_i are the position ,speed and input of ith agent ,respectively .

Definition 1 The multi-agent systems is said to reach consensus when there are some appropriate control inputs u_i make

$$\lim_{t\to\infty}(x_i(t)-x_j(t))=0,\quad \lim_{t\to\infty}\upsilon_i(t)=0,\quad \forall i,j\in \mathrm{V}. \tag{2}$$

C. Preliminaries

Lemma 1 [30] Given any matrix $\Sigma\in ⌋^{2mn}$, a non-zero matrix $\omega\in{}^{2mn}$ satisfies $(\grave{o}_i\otimes\mathbf{1}_{2n})^T\omega=0$, where $i=1,2,\ldots,m$, $m\in{}_{+}$. If and only if the correlation matrix $\bar{\Sigma}\in ⌋^{m(2n-1)}$ of Σ is negative definite, the inequality $\omega^T\Sigma\omega<0$ holds.where $\grave{o}_i\in ⌋^{m}$ means that the i-th element is 1 and the remaining elements are 0 column vectors.

Lemma 2 [31] For a given matrix

$$\Psi=\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}\in R^{n\times n},$$

The following description holds:

1). Ψ has a zero-valued characteristic root and n-1 n-valued characteristic roots.

2). The right eigenvector related to Ψ zero eigenvalue is a column vector $\mathbf{1}_n$, and the left eigenvector related to Ψ zero eigenvalue is a row vector $\mathbf{1}_n^T$.

3). $\Theta_n\in R^{n\times n}$ means that an orthogonal matrix satisfies the following equation

$$\Theta_n^T\Psi\Theta_n=\begin{bmatrix} nI_{n-1} & 0 \\ 0 & 0 \end{bmatrix},$$

$\frac{\mathbf{1}_n}{\sqrt{n}}$ represents the last column of Θ_n . Regarding $\Xi\in R^{n\times n}$ as the Laplacian matrix of any balanced graph, we have

$$\Theta_n^T\Xi\Theta_n=\begin{bmatrix} \vartheta_1 & 0 \\ 0 & 0 \end{bmatrix},\ \vartheta_1\in ⌋^{(n-1)\times(n-1)}.$$

Lemma 3 [32] For a given symmetric positive matrix E, suppose there is a matrix $F\in R^{n\times n}$ such that

$$\begin{bmatrix} E & F \\ * & E \end{bmatrix}\ge 0 ,$$

Then for all constants $\beta\in(0,1)$, we have

$$\begin{bmatrix} \frac{1}{\beta}E & 0 \\ * & \frac{1}{1-\beta}E \end{bmatrix}=\Lambda(\beta)+\begin{bmatrix} E & F \\ * & E \end{bmatrix},$$

where

$$\Lambda(\beta)=\begin{bmatrix} \frac{1}{\beta}E & 0 \\ * & \frac{1}{1-\beta}E \end{bmatrix}-\begin{bmatrix} E & F \\ * & E \end{bmatrix}$$

$$=\begin{bmatrix} \sqrt{\frac{1-\beta}{\beta}}I & 0 \\ * & \sqrt{-\frac{\beta}{1-\beta}}I \end{bmatrix}\begin{bmatrix} E & F \\ * & E \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{\frac{1-\beta}{\beta}}I & 0 \\ * & \sqrt{-\frac{\beta}{1-\beta}}I \end{bmatrix}\ge 0 .$$

Lemma 4 [33]Given a symmetric positive definite matrix $⌋\in R^n$ and any differentiable function $\delta:\left[a,b\right]\to R^n$, the following inequality holds:

$$\int_{\hbar_2}^{\hbar_1}\dot{\delta}^T(s)ℷ\,\dot{\delta}(s)ds \ \ge\ \frac{1}{\hbar_1-\hbar_2}\aleph^T\begin{bmatrix} ℷ & 0 & 0 \\ * & 3ℷ & 0 \\ * & * & 5ℷ \end{bmatrix}\aleph$$

where

$$\aleph = \begin{bmatrix} \delta(\hbar_1)-\delta(\hbar_2) \\ \delta(\hbar_1)+\delta(\hbar_2)-\dfrac{2}{\hbar_1-\hbar_2}\int_{\hbar_2}^{\hbar_1}\delta(s)ds \\ \delta(\hbar_1)-\delta(\hbar_2)-\dfrac{6}{\tau_1}\int_{\hbar_2}^{\hbar_1}[2\dfrac{s-\hbar_2}{\hbar_1-\hbar_2}-1]\delta(s)ds \end{bmatrix}$$

Lemma 5 [34]If there is a symmetric positive definite matrix $\amalg \in R^n$ and any continuous differentiable function $\delta:[a,b]\to R^n$, then the following inequality holds:

$$\int_{\hbar_2}^{\hbar_1}\int_{\theta}^{\hbar_1}\dot{\delta}^T(s)\amalg\dot{\delta}(s)dsd\theta \ \geq$$
$$\frac{2}{(\hbar_1-\hbar_2)^2}\begin{bmatrix}\varpi_1\\ \varpi_2\end{bmatrix}^T\begin{bmatrix}\amalg & 0\\ 0 & \amalg\end{bmatrix}\begin{bmatrix}\varpi_1\\ \varpi_2\end{bmatrix}$$

where

$$\varpi_1=(\hbar_1-\hbar_2)\delta(\hbar_1)-\int_{\hbar_2}^{\hbar_1}\delta(s)ds,$$
$$\varpi_2=\frac{\sqrt{2}}{2}(\hbar_1-\hbar_2)\delta(\hbar_1)+\sqrt{2}\int_{\hbar_2}^{\hbar_1}\delta(s)ds-$$
$$\frac{3\sqrt{2}}{\hbar_1-\hbar_2}\int_{\hbar_2}^{\hbar_1}\int_{\theta}^{\hbar_1}\delta(s)dsd\theta.$$

III. MAIN RESULT

Based on the multi-agent system with time-delay (1), this section constructs a new Lyapunov function. In order to further reduce the conservativeness of the system, triple integral terms are added to the Lyapunov function. Use Wirtinger-based double integral inequality and second- order Bessel- Legendre inequality to effectively scale the double integral term and single integral term generated by the derivation of the Lyapunov function. In the end, a suff- icient condition for consensus with less conservativeness is obtained.

For ease of operation, let

$$\grave{o}_m=\Big[\underbrace{0,\cdots,0}_{m-1},I,\underbrace{0,\cdots,0}_{10-m}\Big]^T,\ m=1,\ 2,\ \ldots,\ 14.$$

$$\xi_3(t)=[\tilde{n}_3^T(t),\ \eta_3^T(t),\ \eta_4^T(t),\ \eta_5^T(t),\ \dot{\delta}^T(t)]^T,\ (3)$$

where

$$\tilde{n}_3(t)=[\delta^T(t),\ \delta^T(t-\tau_1),\ \delta^T(t-\tau(t)),\ \delta^T(t-\tau_2)]^T,$$
$$\eta_3(t)=[\frac{1}{\tau_1}\int_{t-\tau_1}^{t}\delta^T(s)ds,\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}\delta^T(s)ds,\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\gamma}\delta^T(s)ds]^T,$$
$$\eta_4(t)=[\frac{1}{\tau_1}\int_{t-\tau_1}^{t}[2\frac{s-(t-\tau 1)}{\tau 1}-1]\delta^T(s)ds,$$
$$\frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}[2\frac{s-(t-\tau(t))}{\tau(t)-\tau 1}-1]\delta^T(s)ds,$$
$$\frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}[2\frac{s-(t-\tau_2)}{\tau_2-\tau(t)}-1]\delta^T(s)ds]^T,$$
$$\eta_5(t)=[\frac{1}{\tau_{12}}\int_{t-\tau_2}^{t-\tau_1}\delta^T(s)ds,\frac{1}{\tau_1}\int_{t-\tau_1}^{t}\int_{\theta}^{t}\delta^T(s)dsd\theta,\frac{1}{\tau_{12}}\int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\delta^T(s)dsd\theta]^T.$$

Introduce inconsistent dynamics as,

$$\dot{\delta}(t)=\Xi_1\delta(t)-\Xi_2\delta(t-\tau(t)),\qquad(4)$$

The analysis of the consensus of the system (1) is transform- ed into a stability analysis of inconsistent dynamics. Based on Lemma 1, judge the negative definiteness of the deriva- tive of the Lyapunov function. The main results of this section are illustrated by the following theorems.

Theorem 1 If there are constants ν,ρ,τ_1,τ_2 ,matrices $\bar{\Gamma}_{ij},\bar{S}_{1k},\bar{Q}_l,\bar{R}_l\in R^{(2n-1)\times(2n-1)}(i=j=1,2,3,4,5;k=1,2,3;l=1,2)$ and matrices M_{11} and M_{12} so that the following linear matr- ix inequality holds, then, through the consensus protocol

$$u_i(t)=-2\kappa\upsilon_i+\Sigma_{j\in N_i}a_{ij}(t)(x_j(t-\tau(t))-x_i(t-\tau(t))),\ (5)$$

a second-order multi-agent system(1) with time-varying delay can be reached asymptotically convergence.

$$\Phi_3=2\lambda_1+\lambda_2+\lambda_3+\lambda_4+2\lambda_5<0\ (6)$$

Where

$$\lambda_1=[\grave{o}_1\quad \tau_1\grave{o}_5\quad \tau_{12}\grave{o}_{11}\quad \tau_1\grave{o}_{12}\quad \tau_{12}\grave{o}_{13}]$$
$$\times\begin{bmatrix}\bar{\Gamma}_{11}&\bar{\Gamma}_{12}&\bar{\Gamma}_{13}&\bar{\Gamma}_{14}&\bar{\Gamma}_{15}\\ *&\bar{\Gamma}_{22}&\bar{\Gamma}_{23}&\bar{\Gamma}_{24}&\bar{\Gamma}_{25}\\ *&*&\bar{\Gamma}_{33}&\bar{\Gamma}_{34}&\bar{\Gamma}_{35}\\ *&*&*&\bar{\Gamma}_{44}&\bar{\Gamma}_{45}\\ *&*&*&*&\bar{\Gamma}_{55}\end{bmatrix}\begin{bmatrix}\grave{o}_{14}^T\\ \grave{o}_1^T-\grave{o}_2^T\\ \grave{o}_2^T-\grave{o}_4^T\\ \tau_1\grave{o}_1^T-\tau_1\grave{o}_5^T\\ \tau_{12}\grave{o}_1^T-\tau_{12}\grave{o}_{11}^T\end{bmatrix}$$

$$\lambda_2=\grave{o}_1(\bar{S}_{11}+\bar{S}_{12}+\bar{S}_{13})\grave{o}_1^T-\grave{o}_2\bar{S}_{12}\grave{o}_2^T$$
$$-(1-\nu)\grave{o}_3\bar{S}_{12}\grave{o}_3^T-\grave{o}_4\bar{S}_{13}\grave{o}_4^T,$$
$$\lambda_3=\grave{o}_{14}(\tau_1^2\bar{Q}_1+\tau_{12}{}^2\bar{Q}_2)\grave{o}_{14}^T$$
$$-[\grave{o}_1-\grave{o}_2,\quad \grave{o}_1+\grave{o}_2-2\grave{o}_5,\quad \grave{o}_1-\grave{o}_2-6\grave{o}_8]\hat{\bar{Q}}_1\begin{bmatrix}\grave{o}_1^T-\grave{o}_2^T\\ \grave{o}_1^T+\grave{o}_2^T-2\grave{o}_5^T\\ \grave{o}_1^T-\grave{o}_2^T-6\grave{o}_8^T\end{bmatrix}$$
$$-[\grave{o}_2-\grave{o}_3,\quad \grave{o}_2+\grave{o}_3-2\grave{o}_6,\quad \grave{o}_2-\grave{o}_3-6\grave{o}_9,$$
$$\grave{o}_3-\grave{o}_4,\quad \grave{o}_3+\grave{o}_4-2\grave{o}_7,\quad \grave{o}_3-\grave{o}_4-6\grave{o}_{10}]$$
$$\times\begin{bmatrix}\frac{1}{\rho}\hat{\bar{Q}}_2 & 0\\ 0 & \frac{1}{1-\rho}\hat{\bar{Q}}_2\end{bmatrix}\begin{bmatrix}\grave{o}_2^T-\grave{o}_3^T\\ \grave{o}_2^T+\grave{o}_3^T-2\grave{o}_6^T\\ \grave{o}_2^T-\grave{o}_3^T-6\grave{o}_9^T\\ \grave{o}_3^T-\grave{o}_4^T\\ \grave{o}_3^T+\grave{o}_4^T-2\grave{o}_7^T\\ \grave{o}_3^T-\grave{o}_4^T-6\grave{o}_{10}^T\end{bmatrix},$$

$$\lambda_4=\grave{o}_{14}(\frac{\tau_1^4}{4}\bar{R}_1+\frac{\tau_{12}^4}{4}\bar{R}_2)\grave{o}_{14}^T$$
$$-\begin{bmatrix}\tau_1\grave{o}_1-\tau_1\grave{o}_5, & \frac{\sqrt{2}}{2}\tau_1\grave{o}_1+\tau_1\sqrt{2}\grave{o}_5-3\sqrt{2}\grave{o}_{12}\end{bmatrix}$$
$$\times\begin{bmatrix}\bar{R}_1&0\\0&\bar{R}_1\end{bmatrix}\begin{bmatrix}\tau_1\grave{o}_1^T-\tau_1\grave{o}_5^T\\ \frac{\sqrt{2}}{2}\tau_1\grave{o}_1^T+\tau_1\sqrt{2}\grave{o}_5^T-3\sqrt{2}\grave{o}_{12}^T\end{bmatrix}$$
$$-\begin{bmatrix}\tau_{12}\grave{o}_1-\tau_{12}\grave{o}_{11}, & \frac{\sqrt{2}}{2}\tau_{12}\grave{o}_1+\tau_{12}\sqrt{2}\grave{o}_{11}-3\sqrt{2}\grave{o}_{13}\end{bmatrix}$$
$$\times\begin{bmatrix}\bar{R}_1&0\\0&\bar{R}_1\end{bmatrix}\begin{bmatrix}\tau_{12}\grave{o}_1^T-\tau_{12}\grave{o}_{11}^T\\ \frac{\sqrt{2}}{2}\tau_{12}\grave{o}_1^T+\tau_{12}\sqrt{2}\grave{o}_{11}^T-3\sqrt{2}\grave{o}_{13}^T\end{bmatrix},$$
$$\lambda_5=(\varepsilon_1\bar{M}_{11}+\varepsilon_{14}\bar{M}_{12})(-\varepsilon_{14}+\bar{\Xi}_1\varepsilon_1-\bar{\Xi}_2\varepsilon_3),$$
$$\bar{\Xi}_1=\mathrm{H}_1^T(I_n\otimes A)\mathrm{H}_1,\ \bar{\Xi}_2=\mathrm{H}_1^T(L\otimes B)\mathrm{H}_1$$

According to Lemma 2, let H_{1n} and H_{2n} the first 2n-1 columns.

Proof : Based on the inconsistent dynamics (4), a new Lyapunov function number containing triple integral terms is constructed

$$\bar{V}(\delta_t)=\sum_{i=1}^{4}\bar{V}_i(\delta_t), \qquad (8)$$

where,

$$\bar{V}_1(\delta_t)=\begin{bmatrix}\delta(t)\\ \int_{t-\tau_1}^{t}\delta(s)ds\\ \int_{t-\tau_2}^{t-\tau_1}\delta(s)ds\\ \int_{t-\tau_1}^{t}\int_{\theta}^{t}\delta(s)dsd\theta\\ \int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\delta(s)dsd\theta\end{bmatrix}^T \times\begin{bmatrix}\Gamma_{11}&\Gamma_{12}&\Gamma_{13}&\Gamma_{14}&\Gamma_{15}\\ *&\Gamma_{22}&\Gamma_{23}&\Gamma_{24}&\Gamma_{25}\\ *&*&\Gamma_{33}&\Gamma_{34}&\Gamma_{35}\\ *&*&*&\Gamma_{44}&\Gamma_{45}\\ *&*&*&*&\Gamma_{55}\end{bmatrix} \times\begin{bmatrix}\delta(t)\\ \int_{t-\tau_1}^{t}\delta(s)ds\\ \int_{t-\tau_2}^{t-\tau_1}\delta(s)ds\\ \int_{t-\tau_1}^{t}\int_{\theta}^{t}\delta(s)dsd\theta\\ \int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\delta(s)dsd\theta\end{bmatrix},$$

$$\bar{V}_2(\delta_t)=\int_{t-\tau_1}^{t}\delta^T(s)S_{11}\delta(s)ds+ \int_{t-\tau(t)}^{t}\delta^T(s)S_{12}\delta(s)ds+\int_{t-\tau_2}^{t}\delta^T(s)S_{13}\delta(s)ds,$$

$$\bar{V}_3(\delta_t)=\tau_1\int_{-\tau_1}^{0}\int_{t+\theta}^{t}\dot{\delta}^T(s)Q_1\dot{\delta}(s)dsd\theta+ \tau_{12}\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\dot{\delta}^T(s)Q_2\dot{\delta}(s)dsd\theta,$$

$$\bar{V}_4(\delta_t)=\frac{\tau_1^{\,2}}{2}\int_{t-\tau_1}^{t}\int_{\theta}^{t}\int_{u}^{t}\dot{\delta}^T(s)R_1\dot{\delta}(s)dsdud\theta+ \frac{\tau_{12}^{\,2}}{2}\int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\int_{u}^{t}\dot{\delta}^T(s)R_2\dot{\delta}(s)dsdud\theta,$$

where, the matrices $\Gamma_{ij},\ S_{1k},\ Q_l,\ R_l\in R^{2n\times 2n}(i=j=1,\ 2,\ 3,\ 4,\ 5;k=1,\ 2,\ 3;l=1,\ 2)$ are all positive definite matrices.

Taking the derivative of the Lyapunov function in formula (5), we get

$$\dot{\bar{V}}(\delta_t)=\sum_{i=1}^{4}\dot{\bar{V}}_i(\delta_t), (8)$$

$$\dot{\bar{V}}_1(\delta_t)=2\begin{bmatrix}\delta(t)\\ \int_{t-\tau_1}^{t}\delta(s)ds\\ \int_{t-\tau_2}^{t-\tau_1}\delta(s)ds\\ \int_{t-\tau_1}^{t}\int_{\theta}^{t}\delta(s)dsd\theta\\ \int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\delta(s)dsd\theta\end{bmatrix}^T \begin{bmatrix}\Gamma_{11}&\Gamma_{12}&\Gamma_{13}&\Gamma_{14}&\Gamma_{15}\\ *&\Gamma_{22}&\Gamma_{23}&\Gamma_{24}&\Gamma_{25}\\ *&*&\Gamma_{33}&\Gamma_{34}&\Gamma_{35}\\ *&*&*&\Gamma_{44}&\Gamma_{45}\\ *&*&*&*&\Gamma_{55}\end{bmatrix} \begin{bmatrix}\dot{\delta}(t)\\ \delta(t)-\delta(t-\tau_1)\\ \delta(t-\tau_1)-\delta(t-\tau_2)\\ \tau_1\delta(t)-\int_{t-\tau_1}^{t}\delta(s)ds\\ \tau_{12}\delta(t)-\int_{t-\tau_2}^{t-\tau_1}\delta(s)ds\end{bmatrix},$$

$$\begin{aligned}\dot{\bar{V}}_1(\delta_t)&=2\xi_3^T(t)\begin{bmatrix}\varepsilon_1&\tau_1\varepsilon_5&\tau_{12}\varepsilon_{11}&\tau_1\varepsilon_{12}&\tau_{12}\varepsilon_{13}\end{bmatrix} \times\begin{bmatrix}\Gamma_{11}&\Gamma_{12}&\Gamma_{13}&\Gamma_{14}&\Gamma_{15}\\ *&\Gamma_{22}&\Gamma_{23}&\Gamma_{24}&\Gamma_{25}\\ *&*&\Gamma_{33}&\Gamma_{34}&\Gamma_{35}\\ *&*&*&\Gamma_{44}&\Gamma_{45}\\ *&*&*&*&\Gamma_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_{14}^T\\ \varepsilon_1^T-\varepsilon_2^T\\ \varepsilon_2^T-\varepsilon_4^T\\ \tau_1\varepsilon_1^T-\tau_1\varepsilon_5^T\\ \tau_{12}\varepsilon_1^T-\tau_{12}\varepsilon_{11}^T\end{bmatrix}\xi_3(t),\\ &=2\xi_3^T(t)\tilde{\mathrm{H}}\tilde{\mathrm{H}}^{\mathrm{T}} \begin{bmatrix}\varepsilon_1&\tau_1\varepsilon_5&\tau_{12}\varepsilon_{11}&\tau_1\varepsilon_{12}&\tau_{12}\varepsilon_{13}\end{bmatrix} \times\begin{bmatrix}\Gamma_{11}&\Gamma_{12}&\Gamma_{13}&\Gamma_{14}&\Gamma_{15}\\ *&\Gamma_{22}&\Gamma_{23}&\Gamma_{24}&\Gamma_{25}\\ *&*&\Gamma_{33}&\Gamma_{34}&\Gamma_{35}\\ *&*&*&\Gamma_{44}&\Gamma_{45}\\ *&*&*&*&\Gamma_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_{14}^T\\ \varepsilon_1^T-\varepsilon_2^T\\ \varepsilon_2^T-\varepsilon_4^T\\ \tau_1\varepsilon_1^T-\tau_1\varepsilon_5^T\\ \tau_{12}\varepsilon_1^T-\tau_{12}\varepsilon_{11}^T\end{bmatrix}\tilde{\mathrm{H}}\tilde{\mathrm{H}}^{\mathrm{T}}\xi_3(t)\\ &=2\xi_3^T(t)\tilde{\mathrm{H}}\hat{\lambda}_1\tilde{\mathrm{H}}^{\mathrm{T}}\xi_3(t)\end{aligned} \qquad (9)$$

where $\tilde{\mathrm{H}}=diag\{\mathrm{H}_{2n},\mathrm{H}_{2n},\ldots,\mathrm{H}_{2n}\}_{20n\times 20n}$

$$\dot{V}_2(\delta_t) = \delta^T(t)(S_{11}+S_{12}+S_{13})\delta(t)$$
$$-(1-\dot{\tau}(t))\delta^T(t-\tau(t))S_{12}\delta(t-\tau(t))$$
$$-\delta^T(t-\tau_1)S_{12}\delta(t-\tau_1)-\delta^T(t-\tau_2)S_{13}\delta(t-\tau_2)$$

$\dot{\tau}(t) \le \nu$, one has

$$\begin{aligned}
&\dot{V}_2(\delta_t)\\
&\le \delta^T(t)(S_{11}+S_{12}+S_{13})\delta(t)-\\
&(1-\nu)\delta^T(t-\tau(t))S_{12}\delta(t-\tau(t))\\
&-\delta^T(t-\tau_1)S_{12}\delta(t-\tau_1)-\delta^T(t-\tau_2)S_{13}\delta(t-\tau_2)\\
&=\xi_3^T(t)(\varepsilon_1(S_{11}+S_{12}+S_{13})\varepsilon_1^T-\varepsilon_2S_{12}\varepsilon_2^T- \qquad (10)\\
&(1-\nu)\varepsilon_3S_{12}\varepsilon_3^T-\varepsilon_4S_{13}\varepsilon_4^T)\xi_3(t)\\
&=\xi_3^T(t)\tilde{\mathrm{H}}\tilde{\mathrm{H}}^{\mathrm{T}}(\varepsilon_1(S_{11}+S_{12}+S_{13})\varepsilon_1^T-\\
&\varepsilon_2S_{12}\varepsilon_2^T-(1-\nu)\varepsilon_3S_{12}\varepsilon_3^T-\varepsilon_4S_{13}\varepsilon_4^T)\tilde{\mathrm{H}}\tilde{\mathrm{H}}^{\mathrm{T}}\xi_3(t)\\
&=\xi_3^T(t)\tilde{\mathrm{H}}\tilde{\lambda}_2\tilde{\mathrm{H}}^{\mathrm{T}}\xi_3(t),
\end{aligned}$$

$$\begin{aligned}
&\dot{V}_3(\delta_t)\\
&=\tau_1^2\dot{\delta}^T(t)Q_1\dot{\delta}(t)-\tau_1\int_{t-\tau_1}^{t}\dot{\delta}^T(s)Q_1\dot{\delta}(s)ds+\\
&\tau_{12}{}^2\dot{\delta}^T(t)Q_2\dot{\delta}(t)-\tau_{12}\int_{t-\tau_2}^{t-\tau_1}\dot{\delta}^T(s)Q_2\dot{\delta}(s)ds\\
&=\dot{\delta}^T(t)(\tau_1^2Q_1+\tau_{12}{}^2Q_2)\dot{\delta}(t)-\\
&\tau_1\int_{t-\tau_1}^{t}\dot{\delta}^T(s)Q_1\dot{\delta}(s)ds-\tau_{12}\int_{t-\tau_2}^{t-\tau_1}\dot{\delta}^T(s)Q_2\dot{\delta}(s)ds
\end{aligned}$$

Through Lemma 3 and Lemma 4, we can get:

$$-\tau_1\int_{t-\tau_1}^{t}\dot{\delta}^T(s)Q_1\dot{\delta}(s)ds\le-\begin{bmatrix}\wp_1\\\wp_2\\\wp_3\end{bmatrix}^T\hat{Q}_1\begin{bmatrix}\wp_1\\\wp_2\\\wp_3\end{bmatrix},$$

$$-\tau_{12}\int_{t-\tau_2}^{t-\tau_1}\dot{\delta}^T(s)Q_2\dot{\delta}(s)ds$$
$$\le-\begin{bmatrix}\wp_4\\\wp_5\\\wp_6\\\wp_7\\\wp_8\\\wp_9\end{bmatrix}^T\begin{bmatrix}\frac{1}{\rho}\hat{Q}_2 & 0\\ 0 & \frac{1}{1-\rho}\hat{Q}_2\end{bmatrix}\begin{bmatrix}\wp_4\\\wp_5\\\wp_6\\\wp_7\\\wp_8\\\wp_9\end{bmatrix},$$

where

$$\hat{Q}_k=diag\{Q_k,\ 3Q_k,\ 5Q_k\},\forall\ k=1,2,3,$$
$$\rho=\frac{\tau(t)-\tau_1}{\tau_{12}},\forall\rho\in[0,1],$$
$$\wp_1=\delta(t)-\delta(t-\tau_1),$$
$$\wp_2=\delta(t)+\delta(t-\tau_1)-\frac{2}{\tau_1}\int_{t-\tau_1}^{t}\delta(s)ds,$$
$$\wp_3=\delta(t)-\delta(t-\tau_1)-\frac{6}{\tau_1}\int_{t-\tau_1}^{t}[2\frac{s-(t-\tau1)}{\tau_1}-1]\delta(s)ds,$$
$$\wp_4=\delta(t-\tau_1)-\delta(t-\tau(t)),$$
$$\wp_5=\delta(t-\tau_1)+\delta(t-\tau(t))-\frac{2}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}\delta(s)ds,$$
$$\wp_6=\delta(t-\tau_1)-\delta(t-\tau(t))-\frac{6}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1}[2\frac{s-(t-\tau(t))}{\tau(t)-\tau_1}-1]\delta(s)ds,$$
$$\wp_7=\delta(t-\tau(t))-\delta(t-\tau_2),$$
$$\wp_8=\delta(t-\tau(t))+\delta(t-\tau_2)-\frac{2}{\tau_2-\tau(t)}\int_{t-\gamma}^{t-\tau(t)}\delta(s)ds,$$
$$\wp_9=\delta(t-\tau(t))-\delta(t-\tau_2)-\frac{6}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)}[2\frac{s-(t-\tau_2)}{\tau_2-\tau(t)}-1]\delta^T(s)ds.$$

Then,we can get

$$\begin{aligned}
&\dot{V}_3(\delta_t)\\
&\le\xi_3^T(t)\tilde{H}\tilde{H}^T(\varepsilon_{14}(\tau_1^2Q_1+\tau_{12}{}^2Q_2)\varepsilon_{14}^T\\
&-[\varepsilon_1-\varepsilon_2,\ \ \varepsilon_1+\varepsilon_2-2\varepsilon_5,\ \ \varepsilon_1-\varepsilon_2-6\varepsilon_8]\hat{Q}_1\\
&\times\begin{bmatrix}\varepsilon_1^T-\varepsilon_2^T\\\varepsilon_1^T+\varepsilon_2^T-2\varepsilon_5^T\\\varepsilon_1^T-\varepsilon_2^T-6\varepsilon_8^T\end{bmatrix}\\
&-[\varepsilon_2-\varepsilon_3,\ \ \varepsilon_2+\varepsilon_3-2\varepsilon_6,\ \ \varepsilon_2-\varepsilon_3-6\varepsilon_9,\\
&\varepsilon_3-\varepsilon_4,\ \ \varepsilon_3+\varepsilon_4-2\varepsilon_7,\ \ \varepsilon_3-\varepsilon_4-6\varepsilon_{10}]\\
&\times\begin{bmatrix}\frac{1}{\rho}\hat{Q}_2 & 0\\ 0 & \frac{1}{1-\rho}\hat{Q}_2\end{bmatrix}\begin{bmatrix}\varepsilon_2^T-\varepsilon_3^T\\\varepsilon_2^T+\varepsilon_3^T-2\varepsilon_6^T\\\varepsilon_2^T-\varepsilon_3^T-6\varepsilon_9^T\\\varepsilon_3^T-\varepsilon_4^T\\\varepsilon_3^T+\varepsilon_4^T-2\varepsilon_7^T\\\varepsilon_3^T-\varepsilon_4^T-6\varepsilon_{10}^T\end{bmatrix})\tilde{\mathrm{H}}\tilde{\mathrm{H}}^T\xi_3(t) \qquad (11)\\
&=\xi_3^T(t)\tilde{\mathrm{H}}\tilde{\lambda}_3\tilde{\mathrm{H}}^T\xi_3(t),
\end{aligned}$$

$$\begin{aligned}
&\dot{V}_4(\delta_t)=\frac{\tau_1^4}{4}\dot{\delta}^T(t)R_1\dot{\delta}(t)-\frac{\tau_1^2}{2}\int_{t-\tau_1}^{t}\int_{\theta}^{t}\dot{\delta}^T(s)R_1\dot{\delta}(s)dsd\theta\\
&+\frac{\tau_{12}^4}{4}\dot{\delta}^T(t)R_2\dot{\delta}(t)\ -\frac{\tau_{12}^2}{2}\int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\dot{\delta}^T(s)R_2\dot{\delta}(s)dsd\theta\\
&=\dot{\delta}^T(t)(\frac{\tau_1^4}{4}R_1+\frac{\tau_{12}^4}{4}R_2)\dot{\delta}(t)-\frac{\tau_1^2}{2}\int_{t-\tau_1}^{t}\int_{\theta}^{t}\dot{\delta}^T(s)R_1\dot{\delta}(s)dsd\theta\\
&-\frac{\tau_{12}^2}{2}\int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\dot{\delta}^T(s)R_2\dot{\delta}(s)dsd\theta
\end{aligned}$$

According to Lemma 5,

$$-\frac{\tau_1^2}{2}\int_{t-\tau_1}^{t}\int_{\theta}^{t}\dot{\delta}^T(s)R_1\dot{\delta}(s)dsd\theta\le$$
$$-\begin{bmatrix}\wp_{11}\\\wp_{12}\end{bmatrix}^T\hat{Q}_1\begin{bmatrix}\wp_{11}\\\wp_{12}\end{bmatrix},$$

$$-\frac{\tau_{12}^2}{2}\int_{t-\tau_2}^{t-\tau_1}\int_{\theta}^{t-\tau_1}\dot{\delta}^T(s)R_2\dot{\delta}(s)dsd\theta \le$$

$$-\begin{bmatrix}\wp_{13}\\ \wp_{14}\end{bmatrix}^T \hat{Q}_1 \begin{bmatrix}\wp_{13}\\ \wp_{14}\end{bmatrix},$$

$$\wp_{11}=\tau_1\delta(t)-\int_{t-\tau_1}^{t}\delta(s)ds,$$

$$\wp_{12}=\frac{\sqrt{2}}{2}\tau_1\delta(t-\tau_1)+\sqrt{2}\int_{t-\tau_1}^{t}\delta(s)ds-$$

$$\frac{3\sqrt{2}}{\tau_1}\int_{t-\tau_1}^{t}\int_{\theta}^{t}\delta(s)dsd\theta,$$

$$\wp_{13}=\tau_{12}\delta(t)-\int_{t-\tau_2}^{t-\tau_1}\delta(s)ds,$$

$$\wp_{14}=\frac{\sqrt{2}}{2}\tau_{12}\delta(t-\tau_1)+\sqrt{2}\int_{t-\tau_2}^{t-\tau_1}\delta(s)ds$$

$$-\frac{3\sqrt{2}}{\tau_{12}}\int_{t-\tau_2}^{t--\tau_1}\int_{\theta}^{t}\delta(s)dsd\theta.$$

Then,we can get

$$\dot{\bar{V}}_4(\delta_t)$$

$$\le \xi_3^T(t)\tilde{H}\tilde{H}^T(\grave{o}_{14}(\frac{\tau_1^4}{4}R_1+\frac{\tau_{12}^4}{4}R_2)\grave{o}_{14}^T$$

$$-\left[\tau_1\grave{o}_1-\tau_1\grave{o}_5,\quad \frac{\sqrt{2}}{2}\tau_1\grave{o}_1+\tau_1\sqrt{2}\grave{o}_5-3\sqrt{2}\grave{o}_{12}\right]$$

$$\times\begin{bmatrix}R_1 & 0\\ 0 & R_1\end{bmatrix}\begin{bmatrix}\tau_1\grave{o}_1^T-\tau_1\grave{o}_5^T\\ \frac{\sqrt{2}}{2}\tau_1\grave{o}_1^T+\tau_1\sqrt{2}\grave{o}_5^T-3\sqrt{2}\grave{o}_{12}^T\end{bmatrix} \quad (12)$$

$$-\left[\tau_{12}\grave{o}_1-\tau_{12}\grave{o}_{11},\quad \frac{\sqrt{2}}{2}\tau_{12}\grave{o}_1+\tau_{12}\sqrt{2}\grave{o}_{11}-3\sqrt{2}\grave{o}_{13}\right]$$

$$\times\begin{bmatrix}R_1 & 0\\ 0 & R_1\end{bmatrix}\begin{bmatrix}\tau_{12}\grave{o}_1^T-\tau_{12}\grave{o}_{11}^T\\ \frac{\sqrt{2}}{2}\tau_{12}\grave{o}_1^T+\tau_{12}\sqrt{2}\grave{o}_{11}^T-3\sqrt{2}\grave{o}_{13}^T\end{bmatrix})\tilde{H}\tilde{H}^T\xi_3(t)$$

$$=\xi_3^T(t)\tilde{H}\lambda_4\tilde{H}^T\xi_3(t),$$

Similarly, introducing the relaxation matrix sum, there are zero terms as follows:

$$2\left[\delta^T(t)M_{11}+\dot{\delta}^T(t)M_{12}\right]$$
$$\times\left[-\dot{\delta}^T(t)+\Xi_1\delta(t)-\Xi_2\delta^T(t-\tau(t))\right]=0,$$

$$2\xi_3^T(t)[\grave{o}_1M_{11}+\grave{o}_{14}M_{12}][-\grave{o}_{14}+\Xi_1\grave{o}_1-\Xi_2\grave{o}_3]\xi_3(t)$$

$$=2\xi_3^T(t)\tilde{H}\tilde{H}^T[\grave{o}_1M_{11}+\grave{o}_{14}M_{12}]$$

$$\times[-\grave{o}_{14}+\Xi_1\grave{o}_1-\Xi_2\grave{o}_3]\tilde{H}\tilde{H}^T\xi_3(t) \quad (13)$$

$$=2\xi_3^T(t)\tilde{H}\lambda_5\tilde{H}^T\xi_3(t)$$

In formulas (9)-(13), the related definition of symbols $\lambda_1-\lambda_5$ has been given in Theorem 1. It is worth ment- ioning that both matrices Ξ_1, Ξ_2, Γ_{ij}, S_{1k}, Q_l and $R_l(i=j=1,\ 2,\ 3,\ 4,\ 5;\ k=1,2,3;\ l=1,2)$ are balanced matrices. Combining formulas (9)-(13), through Lemma 2, we can get:

$$\dot{\bar{V}}(\delta_t)\le\xi_3^T(t)\Phi_3\xi_3(t),\ \tau(t)\in[\tau_1,\tau_2], (14)$$

where, Φ_3 is defined in Theorem 5. $\sum_{i=1}^{2n}\delta_i(t)=0$, from (9)-(13) and Lemma 2, we can get that for any non-zero vector $\xi_3(t)$, satisfy $(\grave{o}_m\otimes\mathbf{1}_{2n})^T\xi_3(t)=0$ $\xi_3(t)=0$ where, any $m\in\{1,2,\ldots,14\}$, $\grave{o}_m\in\rfloor^{14\times14}$ and the same symbol defined in Lemma 1 are the same when m=14.Inequality (5) is a sufficient condition to satisfy the inequality $\dot{V}(\delta_t)\le\xi_1^T(t)\Phi_1\xi_1(t)<0$ That is to say, if the linear matrix inequality in (5) holds, then for $\tau(t)\in[\tau_1,\tau_2]$, there is an inequality $\dot{\bar{V}}(\delta_t)<0$, thus completing the proof of Theorem 1.

IV. SIMULATION

This section mainly uses a simulation example to verify the validity of the consensus conditions proposed in this paper.Consider a multi-agent network topology composed of 4 notes as shown in Fig.1, each note represents an agent. Based on Theorem 1, the relevant parameters $\nu=0.1$, $\tau_1=0.6$, $\tau_2=1.589$, $\kappa=1$, according to the linear matrix inequality theory, have found a feasible solution through simulation. This assumes a time-varying delay $\tau(t)$ = $t-0.3\cos(t)$. Suppose the initial values of the posit- ion state and speed state of the system (1) are respectively to set $[-1.5, -2.2, 1, 0.8]$ and $[-2, -1.8, 0.5, 1.2]$. Fig.2 and Fig.3 respectively show the agents position trajectory and velocity trajectory of the system (1) using the consensus protocol(5) under the network topology of Fig.1. Fig.4 describes the position state error between agents.

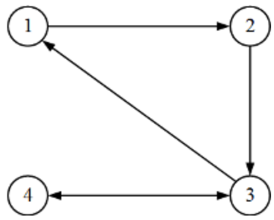

Fig1.Directed communication topology topology of time-delay multi-agent system

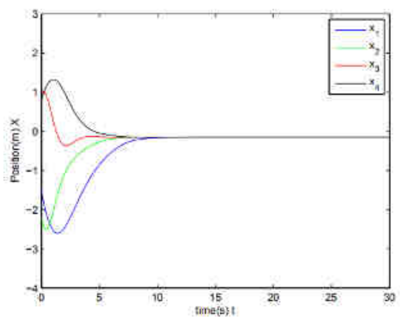

Fig2.Multi-agent position status trajectory

It can be seen from the above simulation result that based on Theorem 1, under the network topology Fig.1, the second-order multi-agent with time-varying delay system consensus problem $\tau(t)$ can be solved. Fig.2 shows when $t\to\infty$, the position states of the four agents reached to the same value, that is to say, although the initial position states of the agents are different, they will eventually tend to be the same under the consensus control protocol.

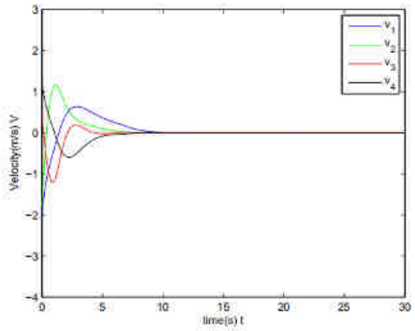

Fig3.Multi-agent speed state trajectory

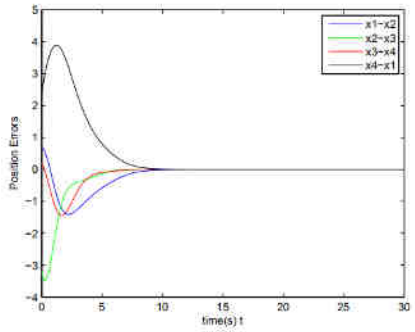

Fig.4 further verifies this result.

When $t \to \infty$, the position state error of any two agents tends to zero, that is, the position states of all agents tend to be the same. It can be seen from Fig.3 that when $t \to \infty$, the speed states of the four agents eventually tend to zero. In summary, the simulation results prove that the multi-agent consensus condition (2) is valid, and the validity of Theorem 1 is verified.

V. Conclusion

Based on the second-order Bessel-Legendre integral inequality, this paper studies the consensus of the second-order multi-agent system with time delay under a directed fixed topology. A Lyapunov function with triple integral is constructed, and the stability of the closed-loop system is analyzed through linear matrix inequality and Lyapunov stability theorem, and sufficient conditions for multi-agent systems to reach agreement are derived. Based on the convex analysis theory and the second-order Bessel-Legendre inequality, low system conservativeness is obtained.

Acknowledgment

The preferred spelling of the word “acknowledgment” in American English is without an “e” after the “g.” Use the singular heading even if you have many acknowledgments. Avoid expressions such as “One of us (S.B.A.) would like to thank ... .” Instead, write “F. A. Author thanks ... .” **Sponsor and financial support acknowledgments are placed in the unnumbered footnote on the first page**.

References

[1] J. Xie, C. C. Liu, Multi-agent systems and their applications, Vol.7, No.1, 188-197, 2017.

[2] M. Dan, I. Herman, M. Sebek, A travelling wave approach to a multi-agent system with a path-graph topology, Systems & Control Letters, Vol.99, No.2, 90-98, 2017.

[3] M. J. Wooldridge, An introduction to multi-agent systems, Wiley & Sons, Vol.4, No.2, 125C128, 2009.

[4] C. W. Reynolds, Flocks, herds and schools: a distributed behavioral model, Computer Graphics, Vol.21, No.4, 25-34, 1987..

[5] Wang Y, Ma Z, Chen G,”Distributed control of cluster lag consensus for first-order multi- agent systems on QUAD vector fields” Journal of the Franklin Institute,vol.355,no.15,pp.7335- 7353,2018.

[6] Li X, Liu J C, Li X G,” Consensus of first-order discrete-time multi-agent systems with time delays” Journal of the Franklin Institute,vol.356,no.10,pp.5315-5331,2019.

[7] Savino H J, Dos Santos C R P, Souza F O,,”Conditions for Consensus of Multi-Agent Systems With Time-Delays and Uncertain Switch- ing Topology” IEEE Transactions on Industrial Electronics,vol.63,no.2,pp.1258-1267.,2016.

[8] Wang A, Mu B, Shi Y,”Event-Triggered Consensus Control for Multiagent Systems With Time- Varying Communication and Event-Detecting Delays” IEEE Transactions on Control Systems Technology,vol.27,no.2,pp.507-515,2019.

[9] Miao G, Ma Q,”Group consensus of the first- order multi-agent systems with nonlinear input constraints” Neurocomputing,pp.113-119,2015.

[10] Chen C L P, Wen G, Liu Y,”Consensus Problems in Networks of Agents with Switching Topology and Time-delays” Adaptive Consensus Control for a Class of Nonlinear Multiagent Time-Delay Sys- tems Using Neural Networks,vol.25,no.6,pp.1217- 1226.,2017.

[11] Ren W, Atkins E,”Distributed multi-vehicle coordinated control via local information exchange” International Journal of Robust Nonlinear Control,vol.17,no.10-11,pp.1002- 1033,2010.

[12] Luo Y, Xiao X, Cao J,”Event-triggered guaranteed cost consensus control for second-order multi- agent systems based on observers” Information ences,vol.546,pp.283-297.,2020.

[13] Mu N, Liao X, Huang T,”Quantized consensus in second-order multi-agent systems via event-trigger control with sampled data” Journal of the Franklin Institute,vol.355,no.5,pp.2720-2734,2018.

[14] Huang N, Duan Z, Chen G,”Some necessary and sufficient conditions for consensus of second-order multi-agent systems with sampled position data” Automatica,vol.63,pp.148-155,2016.

[15] Yang Y, Zhang X, He W,”Sampled-position states based consensus of networked multi-agent systems with second-order dynamics subject to commu- nication delays” Information ences,vol.509,pp.36- 46,2020.

[16] Li X, Chen M Z Q, Su H,”Finite-time con- sensus of second-order multi-agent systems via a structural approach” Journal of the Franklin Institute,vol.353,no.15,3876-3896,2016.

[17] Liu J, Wang C, Cai X,Global finite-time event- triggered consensus for a class of second-order multi-agent systems with the power of positive odd rational number and quantized control inputs” Neurocomputing,vol.360,pp.254-264,2019.

[18] He X, Hao Y, Wang Q.,”Leaderless finite-time consensus for second-order Lipschitz nonlinear multi-agent systems with settling time estima- tion” Physica A: Statal Mechanics and its Applications,vol.514,pp.280-289,2019.

[19] Zhu W, Pu H, Wang D, et al,”Event-based con- sensus of second-order multi-agent systems with discrete time” Automatica,,vol.79,pp.78-83,2017.

[20] Han Y , Li C ,”Second-order Consensus of Discrete-time Multi-agent Systems in Directed Net- works with Nonlinear Dynamics via Impulsive Pro- tocols” Neurocomputing,vol.286,pp.51-57,2018.

[21] Xiang J, Li Y, Wei W, et al,”Synchronisation of linear continuous multi-agent systems with switch- ing topology and communication delay” Con- trol Theory Applications Iet,vol.8,no.14,pp.1415-1420,2014.

[22] Wang D, Zhang N, Wang J, et al,”Cooperative Containment Control of Multiagent Systems Based on Follower Observers With Time Delay” EEE Transactions on Systems Man Cybernetics Systems,vol.47,no.1,pp.13-23.,2017.

[23] Wang Q , Ma Q , Zhou G ,”Distributed PI Control for Consensus of Multi-agent Systems with Time- delay Under Directed Topology” 2019 Chinese Control Conference (CCC),2019.

[24] Ao D, Yang G, Wang X,H Consensus Control of Multi-Agent Systems: A Dynamic Output Feed- back Controller with Time-Delays” 2018 Chinese Control And Decision Conference (CCDC),2018.

[25] Yu S, Yu Z, Jiang H, Hu C,”Leader-following guaranteed performance consensus for second- order multi-agent systems with and without communication delays” IET Control Theory Applica- ions ,vol.12,no.15,pp.2055-2066,2018.

[26] Song Q, Liu F, Wen G, et al,”Distributed Position-Based Consensus of Second-Order Multiagent Systems With Continuous/Intermittent

Communication" IEEE Transactions on Cybernetics,vol.47 ,no.8, pp.1860-1871,2017.

[27] Qian W, Gao Y, Yang Y,"Global Consensus of Multiagent Systems With Internal Delays and Com- munication Delays" IEEE Transactions on Systems Man CyberneticsSystems,vol.49,.no.10 ,pp .1961-1970,2019

[28] Liu K, Seuret A, Xia Y,"Stability analysis of systems with time-varying delays via the second-order Bessel-Legendre inequality" Automatica,vol.76,pp.138-142,2017.

[29] Park M J, Kwon O M, Park J H, et al,"Stability of time-delay systems via Wirtinger-based dou- ble integral inequality" Automatica (Journal of IFAC),vol.55,pp.204-208,2015.

[30] Qin J, Gao H, Zheng W X,"Second-order consensus for multi-agent systems with switching topology and communication delay" International Journal of Robust Nonlinear Control,vol.60,no.6,pp.390-397,2011.

[31] Lin P, Jia Y, Li L,"Distributed robust H consensus control in directed networks of agents with time- delay" Systems Control Letters,vol.57,no.8,pp.643- 653,2008.

[32] Park P G, Ko J W, Jeong C,"Reciprocally convex approach to stability of systems with time-varying delays" Automatica,vol.47,no.1,pp.235-238,2011.

[33] Liu K, Seuret A, Xia Y,"Stability analysis of systems with time-varying delays via the second-order Bessel-Legendre inequality" Automatica,vol.76,pp.138-142,2017.

[34] Seuret A, Gouaisbaut F,"Wirtinger-based integral inequality: Application to time-delay systems" Automatica, vol.49, no.9,pp. 2860-2866,2013.