International Journal of Engineering Research And Management (IJERM)
ISSN: 2349- 2058, Volume-11, Issue-02, February 2024

Stability Analysis for Discrete-Time Stochastic Neural Networks with time-varying delays

Xinran Ding

Abstract— **This article investigates the stability analysis of discrete-time stochastic neural network systems with time-varying delays. In this neural network model, delays are time-varying. By establishing a new set of Lyapunov Krasovskii functionals and applying relevant lemmas, a criterion for robust global exponential stability related to delays in discrete-time stochastic neural network systems with time-varying delays is proposed, and presented in the form of linear matrix inequalities (LMIs), Transform the stability analysis problem to be solved into a feasibility problem for a set of linear matrix inequalities, and finally perform numerical validation using MATLAB to prove the effectiveness of the proposed method.**

Index Terms—**Discrete-time neural networks, Linear matrix inequality, Lyapunov-Krasovskii functional, Robust exponential stability, Stochastic neural networks.**

I. INTRODUCTION

In recent years, research on the theory and application of neural networks has attracted the attention of many scholars at home and abroad. With the continuous deepening and development of research on neural networks, neural networks have been widely applied in many disciplines such as natural language processing, computational optimization, computer vision, biological signal detection, financial risk assessment, etc. [1-6]. The successful application of neural networks in various fields is inseparable from the study of neural network dynamics, including stability, synchronization, periodicity, bifurcation, chaos, etc. The stability of neural network systems plays a crucial prerequisite in practical applications, so research on such problems has received much attention [7-10].

It is worth noting that in the current research on neural networks, the attention to continuous time systems is much higher than that of discrete-time systems. However, in practical engineering applications and production life, compared to continuous time systems, discrete-time neural network systems are more important [11]. Due to the countable nature of discrete systems, precise mathematical models can be used for modeling. In addition, when simulating a continuous time neural network, it is necessary to construct a discrete-time neural network similar to a continuous time neural network [12-15]. Therefore, based on the above reasons, the study of discrete neural network systems has attracted increasing attention from researchers and has achieved some theoretical results [16-19]. Tan et al. [20] derived sufficient conditions for robust exponential stability of discrete quaternion neural networks with time delay and parameter uncertainty using methods such as the compression mapping theorem; Chen et al. [21] proposed an improved cross convex inequality and studied the stability criteria for a time-varying time-delay discrete neural network system; Liu et al. [22] studied the global exponential stability problem of a class of discrete memory recurrent neural networks with time-varying delays. Although there has been an increase in research on discrete neural network systems, there is still insufficient research on continuous neural networks, and there is still a lot of research space on how to obtain lower conservatism for stability criteria of discrete-time neural network systems.

In real neural networks, synaptic transmission is a noisy process caused by random fluctuations in neurotransmitter release and other probabilistic factors. In practical applications, neural network systems are often subject to complex external random disturbances. It is necessary to add a certain degree of randomness in modeling to more accurately describe the system. We can consider it as a type of random input. Random disturbances are also one of the main reasons for the deterioration of system stability. Therefore, the analysis and research on the stability of stochastic neural network systems have important practical significance. Currently, many related literature has been published both domestically and internationally, as shown in [23-26]. To the best of the author's knowledge, most of the system models studied in existing literature are continuous stochastic neural network systems, and there is relatively little research on stability analysis of discrete stochastic neural network systems. This type of problem still deserves further investigation.

The phenomenon of time delay is widely present in various practical systems such as computers, chemical engineering, machinery, aerospace, etc. Any small changes in signal transmission, operating environment, input conditions, etc. in the system will cause the generation of time delay, so the phenomenon of time delay is almost unavoidable. In order to make the model closer to the actual system and better apply the results to practical engineering, researchers have proposed the concept of time-delay systems, which means that the trend of system changes is related to both the current state and the previous state. The existence of time delay is a double-edged sword. On the one hand, in some systems, time delay can be used to improve system stability; On the other hand, in most cases, the existence of time delay is the root cause of the deterioration of system dynamic performance, so studying the stability of time-delay systems has very important practical significance. There are already many theories both domestically and internationally [27-29]. Sun et al. [30] used Brouwer's fixed point theorem, quaternion numerical variation parameters, and other

Manuscript received February 24, 2024
Xinran Ding, School of Software Engineering, Tiangong University, Tianjin, China,

methods to study the existence and stability of equilibrium points in quaternion numerical recurrent neural networks with time-varying and distributed delays; Zhang et al. [31] provided stability criteria for high-order neural networks with unbounded time-varying proportional delays; Mahto et al. [32] provided two delay related stability criteria for neural networks with time-varying delays in the form of linear matrix inequalities. The conclusions of time delay stability can be divided into two types: time-delay related and time-delay independent, with time-delay related having relatively small conservatism. To the best of the author's knowledge, there are few conclusions on the stability analysis of discrete-time stochastic neural network systems with time-varying delays in the existing literature, and it is still an open problem worthy of attention.

Based on the consideration of the above reasons, this paper investigates the stability analysis of discrete-time stochastic neural network systems with time-varying delays. By constructing a new set of Lyapunov Krasovskii functionals, and in order to reduce the conservatism caused by the introduction of Lyapunov Krasovskii functionals, a set of free weighted matrices is also introduced. Through some inequalities Schur's lemma and others derive robust global exponential stability criteria for time-delay related discrete stochastic neural network systems with time-varying delays. The above criteria are given in the form of linear matrix inequalities (LMI), so it is easy to obtain the relevant parameters through the LMI toolbox in MATLAB. The effectiveness of the proposed method is verified through numerical examples.

Notation In this article, R^n represents an n-dimensional Euclidean space; diag{...} represents a block diagonal matrix; the superscript 'T' indicates transpose; the superscript '-1' represents the inverse operation of the corresponding matrix; E[] representing mathematical expectation operators; $\|\cdot\|$ representing the Euclidean vector norm; I representing the identity matrix of appropriate dimensions; $\lambda_{max}(\cdot)$ represents the maximum eigenvalue; the symbol '*' indicates an omitted term caused by symmetry.

II. Problem Formulation and Preliminaries

In a complete probability space $(\Omega,\Gamma,\ P)$, consider a time-varying time-delay discrete stochastic neural network system model with the following form:

$$\begin{cases} x(k+1) = Ax(k) + Bx\big(k-h(k)\big) + Cf\big(x(k)\big) \\ \qquad +Df\Big(x\big(k-h(k)\big)\Big) \\ \qquad +\sigma\big(x(k), x\big(k-h(k)\big), k\big)\omega(k)\big) \\ x(k) = \phi(k), k = [-h_M, 0] \end{cases} \quad (1)$$

Where $x(k) = [x_1(k), x_2(k), x_3(k), \dots, x_n(k)]^T \epsilon R^n$, (k=1,2,...) represents the state vector of the neuron, and $\phi(k)$ is the initial function of the above state vector; Matrices A, B, C, and D are some constant matrices with appropriate dimensions, where $A = diag(a_1,\ a_2, a_3,\ \dots, a_n)$ is a real constant diagonal matrix with appropriate dimensions, represent the state feedback coefficient matrix, constant matrix $C = [c_{ij}]_{n\times n}$ represent the connection weights matrix, constant matrix $D = [d_{ij}]_{n\times n}$ represent the delayed connection weights matrix; $f\big(x(k)\big) = [f_1(x_1(k)), f_2(x_2(k)), f_3(x_3(k)), \dots, f_n(x_n(k))]^T$ and $g\Big(x\big(k-h(k)\big)\Big) = \Big[g_1\Big(x_1\big(k-h(k)\big)\Big), g_2\Big(x_2\big(k-h(k)\big)\Big), g_3\Big(x_3\big(k-h(k)\big)\Big), \dots, g_n\Big(x_n\big(k-h(k)\big)\Big)\Big]^T$ represent the activation function of neurons; The time-varying delay h(k) is a positive integer that satisfies the following conditions:

$$0 < h_m \le h(k) \le h_M \quad (2)$$

where h_m, h_M are known normal number; $\omega(k)$ is a scalar Wiener process on $(\Omega,\Gamma,\ P)$ with

$$E[\omega(k)] = 0, E[\omega^2(k)] = 1, E[\omega(i)\omega(j)] = 0 (i \ne j)$$

We make following assumptions for the neuron network (1).

Assumption 1. $\delta: Z\times R^n \times R^n \to R^n$ satisfying the following assumption:

$$\sigma^T\big(x(k), x\big(k-h(k)\big), k\big)\sigma\big(x(k), x\big(k-h(k)\big), k\big) \le \rho_1 x^T(k)x(k) + \rho_2 x^T\big(k-h(k)\big)x\big(k-h(k)\big) \quad (3)$$

where ρ_1, ρ_2 are two known normal scalar numbers.

Assumption 2. For any i=1, 2, 3, ..., n, and any ξ_1, ξ_2, with known constants c_i^+, c_i^-, such that:

$$c_i^- \le \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \le c_i^+,\ f_i(0) = 0 \quad (4)$$

where $\xi_1 \ne \xi_2$, $c_i^+ c_i^- \ge 0$, $\xi_1, \xi_2 \in R$. The neuron activation functions $f_i(\cdot)$ is continuous and bounded functions.

For the convenience of representation, the following definitions are made in this article:

$$L_1 = \mathrm{diag}(c_1^+c_1^-, c_2^+c_2^-, c_3^+c_3^-, \dots, c_n^+c_n^-)$$

$$L_2 = diag\left(\frac{c_1^+ + c_1^-}{2}, \frac{c_2^+ + c_2^-}{2}, \frac{c_3^+ + c_3^-}{2}, \dots, \frac{c_n^+ + c_n^-}{2}\right)$$

$$L_3 = diag(c_1^-, c_2^-, c_3^-, \dots, c_n^-)$$

$$L_4 = diag(c_1^+ + c_1^-, c_2^+ + c_2^-, c_3^+ + c_3^-, \dots, c_n^+ + c_n^-) \quad (5)$$

The following lemmas and definitions are essential in proving the main theorems.

Lemma 1. (Schur complement) For a given symmetric matrix A,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ * & A_{22} \end{pmatrix}$$

The following three propositions are equivalent:

1) $A < 0$;

2) $A_{11} < 0, A_{22} - A_{12}^T A_{11}^{-1} A_{12} < 0$;

3) $A_{22} < 0, A_{11} - A_{12} A_{22}^{-1} A_{12}^T < 0$;

Lemma 2. [33]. For any integers $b \ge a$, constant matrix $G \in R^{n\times n}$, $G = G^T \ge 0$, vector function $\varphi: \{a, a+1, \dots, b\} \to R^n$, so that we can easy to get that

$$-(b-a+1)\sum_{i=a}^{b} \varphi^T(i) G\,\varphi(i) \le -(\sum_{i=a}^{b}\varphi(i))^T G(\sum_{i=a}^{b}\varphi(i))$$

Definition 1. [34] Consider system (1), if there are constants a>0 and 1<b<0 such that each solution of system (1) satisfies:

$$E\|x(k)\|^2 \le ab^k \sup_{i=[-h_M,0]} E\|x(i)\|^2, \forall k \ge 0$$

The system (1) is said to be robust exponentially stable.

III. MATH

In this section, we investigate the stability analysis of discrete-time stochastic neural network systems with time-varying delays (1) and give a criteria, guaranteeing the robust exponential stability of system (1).

Firstly, we prove the asymptotical stability of system (1).

Theorem 1. Under the condition that Assumptions 1 and 2 hold, for given scalars h_m and h_M , satisfying (2) ,the neural network systems (1) is asymptotical stability, if there exist positive definite matrixs P, Q, S_1,S_2, U_1, U_2, K_1, K_2, any matrices X_{ij} (i=1, 2, 3; j=1, 2, 3, 4) with appropriate dimensions and scalar $\lambda^* > 0$, such that the following LMI hold:

$$\Xi = \begin{bmatrix} \Xi_1 & \Xi_2 \\ * & \Xi_3 \end{bmatrix} < 0 \quad (6)$$

$$P < \lambda^* I \quad (7)$$

where

$$\Xi_1 = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & \theta_{15} & \theta_{16} & \theta_{17} \\ * & \theta_{22} & \theta_{23} & \theta_{24} & \theta_{25} & \theta_{26} & \theta_{27} \\ * & * & \theta_{33} & \theta_{34} & 0 & 0 & 0 \\ * & * & * & \theta_{44} & 0 & 0 & 0 \\ * & * & * & * & \theta_{55} & \theta_{56} & \theta_{57} \\ * & * & * & * & * & \theta_{66} & \theta_{67} \\ * & * & * & * & * & * & \theta_{77} \end{bmatrix}$$

$$\Xi_2 = [\theta_1^T \quad \theta_2^T \quad \theta_3^T \quad \theta_4^T \quad 0 \quad 0 \quad 0]^T$$

$$\Xi_3 = diag(-h_M S_1, -\eta(S_1+S_2), -\eta S_2)$$

$$\theta_{11} = A^{\mathrm{T}}PA - P + (\eta+1)Q + \lambda^*\rho_1 I - 2(\eta+1){L_3}^T K_1 + 2(\eta+1){L_4}^T K_2 + X_{11} + X_{11}^T - L_1 + U_2 - U_1^T$$

$$\theta_{12} = A^{\mathrm{T}}PB - X_{11} + X_{12} + X_{21} - X_{31}$$

$$\theta_{13} = X_{13}^T + X_{31} + U_1$$

$$\theta_{14} = -X_{21} + X_{14}^T$$

$$\theta_{15} = (A-I)^{\mathrm{T}}P$$

$$\theta_{16} = A^{\mathrm{T}}PC + (\eta+1)K_1^T - 2(\eta+1)K_2^T + L_2$$

$$\theta_{17} = A^{\mathrm{T}}PD$$

$$\theta_{22} = B^{\mathrm{T}}PB - Q + 2{L_3}^T K_1 - 2{L_4}^T K_2 - X_{12} - X_{12}^T + X_{22} + X_{22}^T + \lambda^*\rho_2 I - L_1 - X_{32} - X_{32}^T$$

$$\theta_{23} = -X_{13} + X_{23} + X_{32} - X_{33}^T$$

$$\theta_{24} = -X_{22} - X_{34}^T + X_{24}^T - X_{14}^T$$

$$\theta_{25} = B^{\mathrm{T}}P$$

$$\theta_{26} = B^{\mathrm{T}}PC$$

$$\theta_{27} = B^{\mathrm{T}}PD - K_1^T + K_2^T + L_2$$

$$\theta_{33} = X_{33} + X_{33}^T - U_1$$

$$\theta_{34} = -X_{23} + X_{34}^T$$

$$\theta_{44} = -U_2 - X_{24}^T - X_{24}$$

$$\theta_{55} = -2P + h_M S_1 + \eta S_2 + h_m^2 U_1$$

$$\theta_{56} = PC$$

$$\theta_{66} = C^{\mathrm{T}}PC - I$$

$$\theta_{67} = C^{\mathrm{T}}PD$$

$$\theta_{77} = D^{\mathrm{T}}PD - I$$

$$\theta_1 = [h_M X_{11} \quad \eta X_{21} \quad \eta X_{31}]$$

$$\theta_2 = [h_M X_{12} \quad \eta X_{22} \quad \eta X_{32}]$$

$$\theta_3 = [h_M X_{13} \quad \eta X_{23} \quad \eta X_{33}]$$

$$\theta_4 = [h_M X_{14} \quad \eta X_{24} \quad \eta X_{34}]$$

$$X_1 = [X_{11}^T \quad X_{12}^T \quad X_{13}^T \quad X_{14}^T \quad 0 \quad 0 \quad 0]^T$$

$$X_2 = [X_{21}^T \quad X_{22}^T \quad X_{23}^T \quad X_{24}^T \quad 0 \quad 0 \quad 0]^T$$

$$X_3 = [X_{31}^T \quad X_{32}^T \quad X_{33}^T \quad X_{34}^T \quad 0 \quad 0 \quad 0]^T$$

Proof. Construct the following Lyapunov Krasovskii functional for discrete neural network systems with time-varying delays (1):

$$V(k) = \sum_{i=1}^{9} V_i(k) \quad (8)$$

where

$$V_1(k) = x^{\mathrm{T}}(k)Px(k)$$

$$V_2(k) = \sum_{i=k-h(k)}^{k-1} x^{\mathrm{T}}(i)Qx(i)$$

$$V_3(k) = \sum_{i=-h_M+1}^{-h_m} \sum_{j=k+i}^{k-1} x^{\mathrm{T}}(j)Qx(j)$$

$$V_4(k) = \sum_{i=-h_M}^{-1} \sum_{j=k+i}^{k-1} e^{\mathrm{T}}(j)S_1 e(j)$$

$$V_5(k) = \sum_{i=-h_M}^{-h_m-1} \sum_{j=k+i}^{k-1} e^{\mathrm{T}}(j)S_2 e(j)$$

$$V_6(k) = h_m \sum_{i=-h_m}^{-1} \sum_{j=k+i}^{k-1} e^{\mathrm{T}}(j)U_1 e(j)$$

$$V_7(k) = \sum_{i=k-h_M}^{k-1} x^{\mathrm{T}}(i)U_2 x(i)$$

$$V_8(k) = 2\sum_{i=k-h(k)}^{k-1} \left[f(x(i)) - L_3 x(i)\right]^T K_1 x(i) + 2\sum_{i=-h_M+1}^{-h_m} \sum_{j=k+i}^{k-1} \left[f(x(i)) - L_3 x(i)\right]^T K_1 x(i)$$

$$V_9(k) = 2\sum_{i=k-h(k)}^{k-1}[L_4x(i) - f(x(i))]^T K_2x(i) + 2\sum_{i=-h_M+1}^{-h_m}\sum_{j=k+i}^{k-1}[L_4x(i) - f(x(i))]^T K_2x(i)$$

We defined:

$e(k) = x(k+1) - x(k) \qquad (9)$

$\eta = h_M - h_m \qquad (10)$

By performing forward differentiation along the solution of (1), we can obtain:

$$\Delta V(k) = \sum_{i=1}^{7}\Delta V_i(k) \qquad (11)$$

$\Delta V_i(k)$, (i=1,2,3,4,5,6,7), the calculation results are as follows:

$$\begin{aligned}\Delta V_1(k) &= x^{\mathrm{T}}(k+1)Px(k+1) - x^{\mathrm{T}}(k)Px(k)\\ &= [\mathrm{A}x(k) + Bx(k-h(k)) + Cf(x(k)) + Df\left(x(k-h(k))\right)]^{\mathrm{T}}P\left[\mathrm{A}x(k) + Bx(k-h(k)) + Cf(x(k)) + Df\left(x(k-h(k))\right)\right]\\ &\quad + [\sigma(x(k), x(k-h(k)), k)\omega(k)]^{\mathrm{T}}P[\sigma(x(k), x(k-h(k)), k)\omega(k)] - x^{\mathrm{T}}(k)Px(k)\end{aligned}$$

From Assumption 1 and equation (7), it can be concluded that

$$\begin{aligned}&[\sigma(x(k), x(k-h(k)), k)\omega(k)]^{\mathrm{T}}P[\sigma(x(k), x(k-h(k)), k)\omega(k)]\\ &\le \lambda_{max}(P)[\sigma(x(k), x(k-h(k)), k)\omega(k)]^{\mathrm{T}}[\sigma(x(k), x(k-h(k)), k)\omega(k)]\\ &\le \lambda^*\rho_1x^{\mathrm{T}}(k)x(k) + \lambda^*\rho_2x^{\mathrm{T}}(k-h(k))x(k-h(k)),\end{aligned}$$

so that

$$\begin{aligned}\Delta V_1(k) \le{}& [\mathrm{A}x(k) + Bx(k-h(k)) + Cf(x(k)) + Df\left(x(k-h(k))\right)]^{\mathrm{T}}P\left[\mathrm{A}x(k) + Bx(k-h(k)) + Cf(x(k)) + Df\left(x(k-h(k))\right)\right] + \lambda^*\rho_1x^{\mathrm{T}}(k)x(k)\\ &+ \lambda^*\rho_2x^{\mathrm{T}}(k-h(k))x(k-h(k)) - x^{\mathrm{T}}(k)Px(k) \qquad (12)\end{aligned}$$

$$\begin{aligned}\Delta V_2(k) &= \sum_{i=k-h(k+1)+1}^{k}x^{\mathrm{T}}(i)Qx(i) - \sum_{i=k-h(k)}^{k-1}x^{\mathrm{T}}(i)Qx(i)\\ &= x^{\mathrm{T}}(k)Qx(k) + \sum_{i=k-h(k+1)+1}^{k-1}x^{\mathrm{T}}(i)Qx(i) - x^{\mathrm{T}}(k-h(k))Qx(k-h(k)) - \sum_{i=k-h(k)+1}^{k-1}x^{\mathrm{T}}(i)Qx(i)\\ &\le x^{\mathrm{T}}(k)Qx(k) - x^{\mathrm{T}}(k-h(k))Qx(k-h(k)) + \sum_{i=k-h_M+1}^{k-h_m}x^{\mathrm{T}}(i)Qx(i) \qquad (13)\end{aligned}$$

$$\begin{aligned}\Delta V_3(k) &= \sum_{i=-h_M+1}^{-h_m}\sum_{j=k+i+1}^{k}x^{\mathrm{T}}(j)Qx(j) - \sum_{i=-h_M+1}^{-h_m}\sum_{j=k+i}^{k-1}x^{\mathrm{T}}(j)Qx(j)\\ &= \sum_{i=-h_M+1}^{-h_m}[x^{\mathrm{T}}(k)Qx(k) - x^{\mathrm{T}}(k+i)Qx(k+i)]\\ &= \eta x^{\mathrm{T}}(k)Qx(k) - \sum_{i=k-h_M+1}^{k-h_m}x^{\mathrm{T}}(i)Qx(i) \qquad (14)\end{aligned}$$

$$\begin{aligned}\Delta V_4(k) &= \sum_{i=-h_M}^{-1}\sum_{j=k+i+1}^{k}e^{\mathrm{T}}(j)S_1e(j) - \sum_{i=-h_M}^{-1}\sum_{j=k+i}^{k-1}e^{\mathrm{T}}(j)S_1e(j)\\ &= \sum_{i=-h_M}^{-1}[\sum_{j=k+i+1}^{k}e^{\mathrm{T}}(j)S_1e(j) - \sum_{j=k+i}^{k-1}e^{\mathrm{T}}(j)S_1e(j)]\\ &= h_Me^{\mathrm{T}}(k)S_1e(k) - \sum_{i=k-h_M}^{k-1}e^{\mathrm{T}}(i)S_1e(j) \qquad (15)\end{aligned}$$

$$\begin{aligned}\Delta V_5(k) &= \sum_{i=-h_M}^{-h_m-1}\sum_{j=k+i+1}^{k}e^{\mathrm{T}}(j)S_2e(j) - \sum_{i=-h_M}^{-h_m-1}\sum_{j=k+i}^{k-1}e^{\mathrm{T}}(j)S_2e(j)\\ &= \sum_{i=-h_M}^{-h_m-1}[\sum_{j=k+i+1}^{k}e^{\mathrm{T}}(j)S_2e(j) - \sum_{j=k+i}^{k-1}e^{\mathrm{T}}(j)S_2e(j)]\\ &= \eta e^{\mathrm{T}}(k)S_2e(k) - \sum_{i=k-h_M}^{k-h_m-1}e^{\mathrm{T}}(i)S_2e(j) \qquad (16)\end{aligned}$$

For the convenience of subsequent processing, organizing the accumulated part of $\Delta V_4(k)$ and $\Delta V_5(k)$ can obtain:

$$-\sum_{i=k-h_M}^{k-1}e^{\mathrm{T}}(i)S_1e(j) = -[\sum_{i=k-h_M}^{k-h(k)-1}e^{\mathrm{T}}(i)S_1e(j) + \sum_{i=k-h(k)}^{k-1}e^{\mathrm{T}}(i)S_1e(j)] \qquad (17)$$

$$-\sum_{i=k-h_M}^{k-h_m-1}e^{\mathrm{T}}(i)S_2e(j) = -[\sum_{i=k-h_M}^{k-h(k)-1}e^{\mathrm{T}}(i)S_2e(j) + \sum_{i=k-h(k)}^{k-h_m-1}e^{\mathrm{T}}(i)S_2e(j)] \qquad (18)$$

Combine (17) and (18), we can obtain:

$$-\sum_{i=k-h_M}^{k-1}e^{\mathrm{T}}(i)S_1e(j) - \sum_{i=k-h_M}^{k-h_m-1}e^{\mathrm{T}}(i)S_2e(j)$$

$$= -\sum_{i=k-h_M}^{k-h(k)-1} e^{\mathrm{T}}(i)(S_1+S_2)e(j) - \sum_{i=k-h(k)}^{k-1} e^{\mathrm{T}}(i)S_1e(j) - \sum_{i=k-h(k)}^{k-h_m-1} e^{\mathrm{T}}(i)S_2e(j) \quad (19)$$

$$\Delta V_6(k) = h_m \sum_{i=-h_m}^{-1}\sum_{j=k+i+1}^{k} e^{\mathrm{T}}(j)U_1e(j) - h_m \sum_{i=-h_m}^{-1}\sum_{j=k+i}^{k-1} e^{\mathrm{T}}(j)U_1e(j)$$

$$= h_m \sum_{i=-h_m}^{-1} [e^{\mathrm{T}}(k)U_1e(k) - e^{\mathrm{T}}(k+i)U_1e(k+i)]$$

$$= h_m{}^2 e^{\mathrm{T}}(k)U_1e(k) - h_m \sum_{i=k-h_m}^{k-1} e^{\mathrm{T}}(i)U_1e(i)$$

Combine with Lemma 2, we can get:

$$-h_m \sum_{i=k-h_m}^{k-1} e^{\mathrm{T}}(i)U_1e(i) \le -[\sum_{i=k-h_m}^{k-1} e^{\mathrm{T}}(i)]U_1[\sum_{i=k-h_m}^{k-1} e(i)]$$

$$\le -[x(k)-x(k-h_m)]^T U_1[x(k)-x(k-h_m)]$$

so that:

$$\Delta V_6(k) \le h_m{}^2 e^{\mathrm{T}}(k)U_1e(k) - [x(k)-x(k-h_m)]^T U_1[x(k) - x(k-h_m)] \quad (20)$$

$$\Delta V_7(k) = \sum_{i=k-h_M+1}^{k} x^{\mathrm{T}}(i)U_2x(i) - \sum_{i=k-h_M}^{k-1} x^{\mathrm{T}}(i)U_2x(i)$$

$$= x^T(k)U_2x(k) - x^T(k-h_M)U_2x(k-h_M) \quad (21)$$

$$\Delta V_8(k) = 2\sum_{i=k+1-h(k+1)}^{k-1} [f(x(i)) - L_3x(i)]^T K_1x(i) + 2[f(x(k)) - L_3x(k)]^T K_1x(k) + 2\sum_{i=k-h_M+2}^{k+1-h_m}\sum_{j=i}^{k}[f(x(j)) - L_3x(\mathrm{j})]^T K_1x(\mathrm{j}) - 2\sum_{i=k+1-h(k)}^{k-1} [f(x(i)) - L_3x(i)]^T K_1x(i) - 2[f(x(k)) - L_3x(k)]^T K_1x(k) - 2\sum_{i=k-h_M+1}^{k-h_m}\sum_{j=i}^{k-1}[f(x(j)) - L_3x(\mathrm{j})]^T K_1x(\mathrm{j})$$

$$\le 2(\eta+1)[f(x(k)) - L_3x(k)]^T K_1x(k) - 2[f(x(k-h(k))) - L_3x(k-h(k))]^T K_1x(k-h(k)) \quad (22)$$

$$\Delta V_9(k) = 2\sum_{i=k+1-h(k+1)}^{k-1} [L_4x(i) - f(x(i))]^T K_2x(i) + 2[L_4x(i) - f(x(i))]^T K_2x(k) + 2\sum_{i=k-h_M+2}^{k+1-h_m}\sum_{j=i}^{k}[L_4x(i) - f(x(i))]^T K_2x(\mathrm{j}) - 2\sum_{i=k+1-h(k)}^{k-1} [L_4x(i) - f(x(i))]^T K_2x(i) - 2[L_4x(i) - f(x(i))]^T K_2x(k) - 2\sum_{i=k-h_M+1}^{k-h_m}\sum_{j=i}^{k-1}[L_4x(i) - f(x(i))]^T K_2x(\mathrm{j})$$

$$\le 2(\eta+1)[L_4x(\mathrm{k}) - f(x(\mathrm{k}))]^T K_2x(k) - 2[L_4x(k-h(k)) - f(x(k-h(k)))]^T K_2x(k-h(k)) \quad (23)$$

Combine (12)-(23), we can easy to obtain:

$$\Delta V(k) \le [\mathrm{A}x(k) + Bx(k-h(k)) + Cf(x(k)) + Df\left(x(k-h(k))\right)]^{\mathrm{T}} P \left[\mathrm{A}x(k) + Bx(k-h(k)) + Cf(x(k)) + Df\left(x(k-h(k))\right)\right] + x^{\mathrm{T}}(k)[U_2 - P + \lambda^*\rho_1 + (\eta+1)Q - 2(\eta+1)L_3K_1 + 2(\eta+1)L_4^TK_2]x(k) + x^T(k-h(k))(\lambda^*\rho_2 - Q + 2L_3^TK_1 - 2L_4^TK_2)x(k-h(k)) - x^T(k-h_M)U_2x(k-h_M) + e^{\mathrm{T}}(k)(h_MS_1 + \eta S_2 + h_m{}^2U_1)e(k) - [x(k) - x(k-h_m)]^T U_1[x(k) - x(k-h_m)] + f(x(k))[2(\eta+1)K_1 - 2(\eta+1)K_2]x(k) + f(x(k-h(k)))[-2K_1 - 2K_2]x(k-h(k)) - \sum_{i=k-h_M}^{k-h(k)-1} e^{\mathrm{T}}(i)(S_1+S_2)e(j) - \sum_{i=k-h(k)}^{k-1} e^{\mathrm{T}}(i)S_1e(j) - \sum_{i=k-h(k)}^{k-h_m-1} e^{\mathrm{T}}(i)S_2e(j) \quad (24)$$

We defined

$$\alpha^T(k) = [x^{\mathrm{T}}(k) \quad x^{\mathrm{T}}(k-h_m) \quad x^{\mathrm{T}}(k-h(k)) \quad x^{\mathrm{T}}(k-h_M) \quad f^{\mathrm{T}}(x(k)) \quad f^{\mathrm{T}}\left(x(k-h(k))\right)]$$

According to the definition of e (k), it can be inferred that:

$$0 = x(k) - x(k-h(k)) - \sum_{i=k-h(k)}^{k-1} e(i) \quad (25)$$

$$0 = x(k-h(k)) - x(k-h_M) - \sum_{i=k-h_M}^{k-h(k)-1} e(i) \quad (26)$$

$$0 = x(k-h_m) - x\big(k-h(k)\big) - \sum_{i=k-h(k)}^{k-h_m-1} e(i) \quad (27)$$

$$0 = (A-I)x(k) + Bx\big(k-h(k)\big) + Cf\big(x(k)\big) + Df\left(x\big(k-h(k)\big)\right) - e(k) \quad (28)$$

Combine (25)-(28), for any matrix of appropriate dimensions X_{ij} (i=1, 2, 3 ; j=1, 2, 3, 4), the following equation always holds:

$$2[x^T(k)X_{11} + x^T(k-h_m)X_{12} + x^T(k-h(k))X_{13} + x^T(k-h_M)X_{14}] \times [x(k) - x\big(k-h(k)\big) - \sum_{i=k-h(k)}^{k-1} e(i)] = 0 \quad (29)$$

$$2[x^T(k)X_{21} + x^T(k-h_m)X_{22} + x^T(k-h(k))X_{23} + x^T(k-h_M)X_{24}] \times [x\big(k-h(k)\big) - x(k-h_M) - \sum_{i=k-h_M}^{k-h(k)-1} e(i)] = 0 \quad (30)$$

$$2[x^T(k)X_{31} + x^T(k-h_m)X_{32} + x^T(k-h(k))X_{33} + x^T(k-h_M)X_{34}] \times [x(k-h_m) - x\big(k-h(k)\big) - \sum_{i=k-h(k)}^{k-h_m-1} e(i)] = 0 \quad (31)$$

$$2e^T(k)P[(A-I)x(k) + Bx\big(k-h(k)\big) + Cf\big(x(k)\big) + Df\left(x\big(k-h(k)\big)\right) - e(k)] = 0 \quad (32)$$

From Assumption 2, we can easily obtain follows with i=1, 2, 3, ..., n:

$(f_i(x_i(k)) - c_i^- x_i(k))^T (f_i(x_i(k)) - c_i^+ x_i(k)) \le 0$

So that:

$$\sum_{i=i}^{n} (f_i(x_i(k)) - c_i^- x_i(k))^T (f_i(x_i(k)) - c_i^+ x_i(k)) \le 0 \quad (33)$$

Similarly, it can be inferred that:

$$\sum_{i=i}^{n} (f_i(x_i(k-h(k))) - c_i^- x_i(k-h(k)))^T (f_i(x_i(k-h(k))) - c_i^+ x_i(k-h(k))) \le 0 \quad (34)$$

Combining (33) and (34):

$$0 \le -\sum_{i=i}^{n} (f_i(x_i(k)) - c_i^- x_i(k))^T (f_i(x_i(k)) - c_i^+ x_i(k)) - \sum_{i=i}^{n} (f_i(x_i(k-h(k))) - c_i^- x_i(k-h(k)))^T (f_i(x_i(k-h(k))) - c_i^+ x_i(k-h(k)))$$

$$0 \le -x^T(k)L_1x(k) - f^T(x(k))f(x(k)) + 2x^T(k)L_2f(x(k)) - x^T(k-h(k))L_1x(k-h(k)) - f^T(k-h(k))f(k-h(k)) + 2x^T(k-h(k))L_2f(x(k-h(k))) \quad (35)$$

According to (24) - (32) and (35), the following inequality holds:

$$\Delta V(k) \le \alpha^T(k)[\Xi_1 + h_M X_1 S_1^{-1} X_1^T + \eta X_2 (S_1+S_2)^{-1} X_2^T + \eta X_3 S_2^{-1} X_3^T]\alpha(k) - \sum_{i=k-h(k)}^{k-1} (X_1^T\alpha(k) + S_1 e(i))^T S_1^{-1} (X_1^T\alpha(k) + S_1 e(i)) - \sum_{i=k-h_M}^{k-h(k)-1} (X_2^T\alpha(k) + (S_1+S_2)e(i))^T (S_1+S_2)^{-1} (X_2^T\alpha(k) + (S_1+S_2)e(i)) - \sum_{i=k-h(k)}^{k-h_m-1} (X_3^T\alpha(k) + S_2 e(i))^T S_2^{-1} (X_3^T\alpha(k) + S_2 e(i)) \quad (36)$$

According to Lemma 1, equation (6) is equivalent to:

$$\Xi_1 + h_M X_1 S_1^{-1} X_1^T + \eta X_2 (S_1+S_2)^{-1} X_2^T + \eta X_3 S_2^{-1} X_3^T < 0 \quad (37)$$

Therefore, there exists a sufficiently small scalar $\varepsilon < 0$ that:

$$\Delta V \le -\varepsilon \|x(k)\|^2 < 0 \quad (38)$$

This means that the discrete-time stochastic neural network system with time-varying delays (1) is asymptotically stable.

Next, we will further proof the robust exponential stability of system (1).

According to the definition of V(k) (8), it is easy to obtain:

$$V(k) \leq \lambda_{max}(P)\|x(k)\|^2 + q_1 \sum_{j=k-h_M}^{k-1} \|x(j)\|^2 + q_2 \sum_{j=k-h_M}^{k-1} \|x(j+1)\|^2 \quad (39)$$

where

$$q_1 = \lambda_{max}(U_1) + (\eta+1)\lambda_{max}(Q) + 2h_M\lambda_{max}(S_1+S_2) + 2h_m^2\lambda_{max}(U_2) + 2(\eta+1)[\lambda_{max}(L_3K_1) + \lambda_{max}(L_4K_2)]$$

$$q_2 = 2h_M\lambda_{max}(S_1+S_2) + 2h_m^2\lambda_{max}(U_2)$$

For any $\gamma > 1$, combine equations (38) and (39), it can be obtained that:

$$\gamma^{i+1}V(i+1) - \gamma^i V(i) = \gamma^{i+1}\Delta V(i) + \gamma^i(\gamma-1)V(i) \leq [-\varepsilon\gamma + (\gamma-1)\lambda_{max}(P)]\gamma^i\|x(i)\|^2 + (\gamma-1)[q_1\gamma^i \sum_{j=i-h_M}^{i-1} \|x(j)\|^2 + q_2\gamma^i \sum_{j=i-h_M}^{i-1} \|x(j+1)\|^2] \quad (40)$$

Existence of any integer $T \geq h_M + 1$, accumulate from 0 to T-1 on both sides of the above equation to obtain

$$\gamma^T V(T) - V(0) \leq [-\varepsilon\gamma + (\gamma-1)\lambda_{max}(P)] \sum_{i=0}^{T-1} \gamma^i\|x(i)\|^2 + (\gamma-1)[q_1 \sum_{i=0}^{T-1}\sum_{j=i-h_M}^{i-1} \gamma^i\|x(j)\|^2 + q_2 \sum_{i=0}^{T-1}\sum_{j=i-h_M}^{i-1} \gamma^i\|x(j+1)\|^2] \quad (41)$$

According to [36]. , the two cumulative terms in equation (41) can be calculated separately to obtain:

$$\sum_{i=0}^{T-1}\sum_{j=i-h_M}^{i-1} \gamma^i\|x(j)\|^2 \leq \left(\sum_{j=-h_M}^{-1}\sum_{i=0}^{j+h_M} + \sum_{j=0}^{T-1+h_M}\sum_{i=j+1}^{j+h_M} + \sum_{j=T-h_M}^{T-1}\sum_{i=j+1}^{T-1}\right)\gamma^i\|x(j)\|^2$$

$$\leq h_M^2\gamma^{h_M} \sup_{j=[-h_M,0]} \|x(j)\|^2 + h_M\gamma^{h_M} \sum_{j=0}^{T-1} \gamma^j\|x(j)\|^2 \quad (42)$$

$$\sum_{i=0}^{T-1}\sum_{j=i-h_M}^{i-1} \gamma^i\|x(j+1)\|^2 \leq h_M^2\gamma^{h_M} \sup_{j=[-h_M,0]} \|x(j)\|^2 + h_M\gamma^{h_M} \sum_{j=1}^{T} \gamma^j\|x(j)\|^2 \quad (43)$$

From equation (39), it can be concluded that

$$V(0) \leq [\lambda_{max}(P) + h_M(q_1+q_2)]\, sup_{j=[-h_M,0]}\|x(j)\|^2 \quad (44)$$

By combining equations (39) - (43), it can be concluded that

$$\gamma^T V(T) \leq \beta_1(\gamma) \sum_{j=0}^{T} \gamma^j\|x(j)\|^2 + \beta_2(\gamma) \sup_{j=[-h_M,0]} \|x(j)\|^2 \quad (45)$$

where

$$\beta_1(\gamma) = (\gamma-1)\lambda_{max}(P) - \gamma\varepsilon + (q_1+q_2)(\gamma-1)h_M\gamma^{h_M}$$

$$\beta_2(\gamma) = \lambda_{max}(P) + h_M(q_1+q_2) + (q_1+q_2)(\gamma-1)h_M^2\gamma^{h_M}$$

In addtion, for $\beta_1(1) = -\varepsilon < 0$, there must exist a positive scalar $\gamma_0 > 1$ that makes $\beta_1(\gamma_0) < 0$, thus

$$V(T) \leq \beta_2(\gamma_0)\left(\frac{1}{\gamma_0}\right)^T sup_{j=[-h_M,0]}\|x(j)\|^2 \quad (46)$$

From the definition of V (k), it can be obtained that

$$V(T) \geq \lambda_{min}(P)\|x(T)\|^2 \quad (47)$$

Combining (45) - (47), it can be concluded that:

$$\|x(T)\|^2 \leq \frac{\beta_2(\gamma)}{\lambda_{min}(P)}\left(\frac{1}{\gamma_0}\right)^T sup_{j=[-h_M,0]}\|x(j)\|^2 \quad (48)$$

This means that the discrete-time stochastic neural network system with time-varying delays (1) is robust exponentially stable.

IV. NUMERICAL EXAMPLE

In this section, two numerical examples are provided to demonstrate the effectiveness of the proposed criterion for discrete-time stochastic neural network systems with time-varying delays (1).

Example 1. Consider a neural network system (1) with the following parameters

$$A=\begin{bmatrix}0.4 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.45\end{bmatrix} \quad B=\begin{bmatrix}0.02 & 0 & 0\\ 0 & 0.03 & 0.1\\ 0 & 0.055 & -0.1\end{bmatrix}$$

$$C=\begin{bmatrix}0.4 & 0.25 & 0\\ 0 & -0.015 & -0.02\\ 0.1 & 0.02 & 0.01\end{bmatrix} \quad D=\begin{bmatrix}0 & 0.2 & 0.01\\ 0.2 & 0.01 & 0.1\\ 0.2 & -0.4 & 0.03\end{bmatrix}$$

$$I=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad \rho_1 = 0.01 \quad \rho_2 = 0.02$$

The activation function is described as:

$f_1(s) = tanh(0.6s), \quad f_2(s) = tanh(-0.4s),$
$f_3(s) = tanh(-0.2s);$

Based on the above parameters, it is easy to obtain:

$$L_1 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} L_2 = \begin{bmatrix}0.3 & 0 & 0\\0 & -0.2 & 0\\0 & 0 & -0.1\end{bmatrix}$$

$$L_3 = \begin{bmatrix}0 & 0 & 0\\0 & -0.4 & 0\\0 & 0 & -0.2\end{bmatrix} L_4 = \begin{bmatrix}0.6 & 0 & 0\\0 & -0.4 & 0\\0 & 0 & -0.2\end{bmatrix}$$

By using the LMI toolbox in MATLAB, the following feasible solutions can be obtained by solving (6) and (7), a more details about h_M given in Table 1. :

$$P = \begin{bmatrix}2.1181 & 0.0844 & 0.2751\\0.0844 & 2.2083 & -0.2225\\0.2751 & -0.2225 & 1.2859\end{bmatrix}$$

$$Q = \begin{bmatrix}0.0237 & 0.0152 & 0.0178\\0.0152 & 0.1476 & -0.0516\\0.0178 & -0.0516 & 0.0953\end{bmatrix}$$

$$S_1 = \begin{bmatrix}0.0858 & 0.0040 & 0.0231\\0.0040 & 0.0525 & 0.0132\\0.0231 & 0.0132 & 0.0516\end{bmatrix}$$

$$S_2 = \begin{bmatrix}0.1529 & 0.0013 & 0.0353\\0.0013 & 0.0663 & 0.0177\\0.0353 & 0.0177 & 0.0676\end{bmatrix}$$

$$U_1 = \begin{bmatrix}0.0961 & -0.0050 & 0.0197\\-0.0050 & 0.0253 & 0.0066\\0.0197 & 0.0066 & 0.0261\end{bmatrix}$$

Table 1. Calculated upper bound h_M for given h_m.

h_m	2	3	6	10	15
Theorem 1.	8	9	11	15	20

$$U_2 = \begin{bmatrix}0.1441 & -0.0076 & 0.0325\\-0.0076 & 0.0496 & 0.0102\\0.0325 & 0.0102 & 0.0408\end{bmatrix}$$

$$K_1 = \begin{bmatrix}0.1876 & -0.0059 & -0.0154\\-0.0059 & 0.0418 & 0.0515\\-0.0154 & 0.0515 & 0.0753\end{bmatrix}$$

$$K_2 = \begin{bmatrix}0.0950 & -0.0029 & -0.0075\\-0.0029 & 0.0273 & 0.0228\\-0.0075 & 0.0228 & 0.0345\end{bmatrix}$$

$$\lambda^* = 2.3026$$

Therefore, according to Theorem 1, a neural network system (1) with given parameters is robust globally exponentially stable.

V. CONCLUSION

This paper investigates the stability problem of discrete stochastic neural network systems with time-varying delays. By constructing a new set of Lyapunov Krasovskii functionals, sufficient conditions for robust global exponential stability related to delays in discrete stochastic neural network systems with time-varying delays are proposed, and presented in the form of linear matrix inequalities (LMIs). Finally, the effectiveness of the proposed method is verified through numerical examples.

REFERENCES

[1] C. R. Wang, Y. He, C. K. Zhang, M. Wu, "Stability Analysis of Discrete-Time Neural Networks With a Time-Varying Delay: Extended Free-Weighting Matrices Zero Equation Approach," IEEE TRANSACTIONS ON CYBERNETICS, 2022, DOI:10.1109/TCYB.2022.3201686.

[2] Y. G. Chen, N. N. Zhang, J. J. Yang, "A survey of recent advances on stability analysis, state estimation and synchronization control for neural networks," Neurocomputing, vol. 515, Oct. 2022, pp. 26-36.

[3] X. M. Zhang, Q. L. Han, X. H. Ge, "An overview of neuronal state estimation of neural networks with time-varying delays. Information Sciences," vol. 478, Apr. 2019, pp. 83–99.

[4] C. X. Huang, B. W. Liu, "New studies on dynamic analysis of inertial neural networks involving non-reduced order method. Neurocomputing,"vol. 325, Jan. 2019, pp. 283–287.

[5] X. X. You, S. Y. Dian, R. Guo, S. C. Li, "Exponential stability analysis for discrete-time quaternion-valued neural networks with leakage delay and discrete time-varying delays," Neurocomputing, vol. 430, Jan. 2021, pp. 71-81.

[6] G. Rajchakit, P. Chanthorn, M. Niezabitowski, R. Raja, D. Baleanu, A. Pratap, "Impulsive effects on stability and passivity analysis of memristor-based fractional-order competitive neural networks," Neurocomputing, vol. 417, Dec. 2020, pp. 290–301.

[7] J. R. Zhang, J. G. Lu, X. C. Jin, X. Y. Yang, "Novel results on asymptotic stability and synchronization of fractional-order memristive neural networks with time delays: The 0< δ ≤1 case," Neural Networks, vol. 167, Sep. 2023, pp. 680-691.

[8] T. W. Zhang, Y. K. Li, "Global exponential stability of discrete-time almost automorphic Caputo-Fabrizio BAM fuzzy neural networks via exponential Euler technique," Knowledge-Based Systems, vol. 246, Apr. 2022. DOI: 10.1016/j.knosys.2022.108675.

[9] C. J. Xu, D. Mu, Y. L. Pan, C. Aouiti, Y. C. Pang, L. Y. Yao, "Probing into bifurcation for fractional-order BAM neural networks concerning multiple time delays," Journal of Computaional Science, vol. 62, May. 2022. DOI: 10.1016/j.jocs.2022.101701.

[10] S. L. Chen, H. L. Li, H. B. Bao, L. Zhang, H. J. Jiang, Z. M. Li, "Global Mittag-Leffler stability and synchronization of discrete-time fractional-order delayed quaternion-valued neural networks," Neurocomputing, vol. 511, Sep. 2022, pp. 290-298.

[11] M. Hernandez-Gonzalez, E. A. Hernandez-Vargas, "Discrete-time super-twisting controller using neural networks," Neurocomputing, vol. 447, Apr. 2021, pp. 235-243.

[12] R. J. Yang, B. Wu, Y. Liu, "A Halanay-type inequality approach to the stability analysis of discrete-time neural networks with delays," Applied Mathematics and Computation, vol. 265, Aug. 2015, pp. 696 –707.

[13] L. Li, W. S. Chen, "Exponential stability analysis of quaternion-valued neural networks with proportional delays and linear threshold neurons: Continuous-time and discrete-time cases," Neurocomputing, vol. 381, Mar. 2020, pp.152-166.

[14] S. L. Wu, K. Li, J. Zhang, "Exponential stability of discrete-time neural networks with delay and impulses," Applied Mathematics and Computation, vol.218 (12), Feb. 2012, pp. 6972–6986.

[15] C. K. Zhang, Y. He, L. Jiang, Q. G. Wang, M. Wu, "Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality," IEEE Transactions on Cybernetics, vol. 47 (10), Oct. 2017, pp. 3040–3049.

[16] A. Pratap, R. Raja, J. D. Cao, C. X. Huang, M. Niezabitowski, Bagdasar, O. "Stability of discrete-time fractional-order time-delayed neural networks in complex field," Mathmatical Methods in the Applied Sciences, vol. 44, Jul. 2020, pp. 419-440.

[17] B. Yang, M. N. Hao, M. Han, X. D. Zhao, G. D. Zong, "Exponential Stability of Discrete-Time Neural Networks With Large Delay," IEEE Transactions on Cybernetics, vol. 51(5), May. 2021, pp. 2824-2834.

[18] T. Cai, P. Cheng, F. Q. Yao, M. A. Hua, "Robust exponential stability of discrete-time uncertain impulsive stochastic neural networks with delayed impulses." Neural Networks, vol. 160, Mar. 2023, pp. 227-237.

[19] E. Suntonsinsoungvon, S. Udpin, "Exponential stability of discrete-time uncertain neural networks with multiple time-varying leakage

delays" Mathematics and Computers in Simulation, vol. 171, May. 2020, pp. 233-245.

[20] Y. S. Tan, X. D. Wang, J. Yang, J. Hu, "Robust Exponential Stability for Discrete-Time Quaternion-Valued Neural Networks with Time Delays and Parameter Uncertainties," Neural Processing Letters, vol. 51(3), Feb. 2020, pp. 2317-2335.

[21] Q. Chen, X. G. Liu, P. Y. Guo, H. Liu, X. Y. Li, "Improved results on stability and H∞ performance analysis for discrete-time neural networks with time-varying delay," Computational & Applied Mathmatics, vol. 41(5), Jul. 2022. DOI: 10.1007/s40314-022-01902-6.

[22] H. J. Liu, Z. D. Wang, B. Shen, "Discrete-Time Memristive Recurrent Neural Networks with Time-Varying Delays: Exponential Stability Analysis," In Proceedingsof the 35th Chinese Control Conference 2016, Chengdu, China, 27-29 July 2016; pp. 3584-3589.

[23] L. C. Feng, J. D. Cao, L. Liu, "Robust analysis of discrete time noises for stochastic systems and application in neural networks," International Journal of Control, vol. 93(12), Dec. 2020, pp. 2908-2921.

[24] Q. H. Zhou, L. Wan, H. B. Fu, Q. J. Zhang, "Exponential stability of stochastic Hopfield neural network with mixed multiple delays," Aims Mathematics, vol. 6(4), 2021, pp. 4142-4155.

[25] P. Chanthorn, G. Rajchakit, J. Thipcha, C. Emharuethai, R. Sriraman, C.P. Lim, et al. "Robust Stability of Complex-Valued Stochastic Neural Networks with Time-Varying Delays and Parameter Uncertainties," Mathematics, vol. 8(5), May. 2020. DOI: 10.3390/math8050742.

[26] Y. Cao, R. Sriraman, N. Shyamsundarraj, R. Samidurai, "Robust stability of uncertain stochastic complex-valued neural networks with additive time-varying delays," Mathematics and Computers in Simulation, vol.171, May. 2020, pp. 207-220.

[27] S. Arik, "New Criteria for Stability of Neutral-Type Neural Networks With Multiple Time Delays," IEEE Transactions on Neural Networks and Learning Systems, vol. 31(5), May. 2020, pp. 1504-1513.

[28] O. Faydasicok, "New criteria for global stability of neutral-type Cohen-Grossberg neural networks with multiple delays," Neural Networks, vol. 125, May. 2020, pp. 330-337.

[29] L. Q. Zhou, Z. X. Zhao, "Asymptotic Stability and Polynomial Stability of Impulsive Cohen-Grossberg Neural Networks with Multi-proportional Delays," Neural Processing Letters, vol. 51(3), Jun. 2020, pp. 2607-2627.

[30] J. W. Zhu, J. T. Sun, "Stability of Quaternion-Valued Neural Networks with Mixed Delays," Neural Processing Letters, vol. 49(2), Apr. 2019, pp. 819-833.

[31] W. Q. Shen, X. Zhang, Y. T. Wang, "Stability analysis of high order neural networks with proportional delays," Neurocoputing, vol. 372, Jan. 2020, pp. 33-39.

[32] S. C. Mahto, S. Ghosh, R. K. Saket, S. K. Nagar, "Stability analysis of delayed neural network using new delay-product based functionals" Neurocoputing, vol. 417, Dec. 2020, pp. 106-113.

[33] Y. R. Liu, Z. D. Wang, X. H. Liu, "Robust stability of discrete-time stochastic neural networks with time-varying delays," Neurocoputing, vol. 71(4-6), Jan. 2008, pp. 823-833.

[34] J. J. Yu, K. J. Zhang, S. M. Fei, "Exponential stability criteria for discrete-time recurrent neural networks with time-varying delay," Nonlinear Anal Real World Appl, vol. 11, Feb. 2010, pp. 207–216.

Xinran Ding She received the B.Eng degree in Software Engineering from the Tianjin University of Technology, Tianjin, China, in 2021.